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NUMERICAL SOLUTION OF A TWO-DIMENSIONAL PARABOLIC TRANSMISSION PROBLEM

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Abstract. An initial boundary value problem for a two-dimensional parabolic equation in two disconnected rectangles is investigated. A finite difference scheme approximating this problem is proposed and analyzed. An estimate of the convergence rate, compatible with the smoothness of the input data (up to a logarithmic factor of the mesh-size), is obtained.

Key words. transmission problem, initial-boundary value problem, disconnected domains, Sobolev spaces, finite differences, convergence rate.

1. Introduction and formulation of the problem

Layers with material properties which significantly differ from those of the surrounding medium appear in a variety of applications. The layer may have a structural role (as in the case of glue), a thermal role (as in the case of a thin thermal insulator), an electromagnetic or optical role etc. Mathematical models of energy and mass transfer in domains with layers lead to so called interface or transmission problems. For example, in [11] we investigated the heat transfer process in the presence of thin layer with concentrated capacity.

In this paper we focus our attention to transmission problems whose solutions are defined in two (or more) disconnected domains. For example, such a situation occurs when the solution in the intermediate region is known, or can be determined from a simpler equation. The effect of the intermediate region can be modelled (see [4, 5, 6, 7, 16, 19, 22]) by means of nonlocal jump conditions across the intermediate region.

In [12, 14] we considered transmission problem for one-dimensional parabolic equation in two disconnected intervals. After an analysis of the strong and weak solutions in specific Sobolev-like spaces, difference schemes for its approximation are constructed and investigated for convergence. Also, one-dimensional elliptic and hyperbolic problems was studied in [17] and [13], respectively. Analytical properties two-dimensional parabolic problem in two disconnected rectangles are investigated in [15]. Here we propose finite difference schemes for its numerical solution.

As a model example, we consider the following initial-boundary-value problem (IBVP): Find functions $u_1(x,y,t)$ and $u_2(x,y,t)$ that satisfy the system of parabolic equations

$$(1)\quad \frac{\partial u_1}{\partial t} - \frac{\partial}{\partial x}\left(p_1(x,y)\frac{\partial u_1}{\partial x}\right) - \frac{\partial}{\partial y}\left(q_1(x,y)\frac{\partial u_1}{\partial y}\right) + r_1(x,y)u_1 = f_1(x,y,t),$$
$$(x,y)\in\Omega_1\equiv(a_1,b_1)\times(c,d),\quad t>0,$$

$$(2)\quad \frac{\partial u_2}{\partial t} - \frac{\partial}{\partial x}\left(p_2(x,y)\frac{\partial u_2}{\partial x}\right) - \frac{\partial}{\partial y}\left(q_2(x,y)\frac{\partial u_2}{\partial y}\right) + r_2(x,y)u_2 = f_2(x,y,t),$$
$$(x,y)\in\Omega_2\equiv(a_2,b_2)\times(c,d),\quad t>0,$$

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where $-\infty < a_1 < b_1 < a_2 < b_2 < +\infty$, the internal conjugation conditions of non-local Robin-Dirichlet type

$$p_1(b_1,y)\frac{\partial u_1}{\partial x}(b_1,y,t) + \alpha_1(y)u_1(b_1,y,t) = \int_c^d \beta_1(y,y_*)u_2(a_2,y_*,t)\,\mathrm{d}y_*, \tag{3}$$

$$-p_2(a_2,y)\frac{\partial u_2}{\partial x}(a_2,y,t) + \alpha_2(y)u_2(a_2,y,t) = \int_c^d \beta_2(y,y_*)u_1(b_1,y_*,t)\,\mathrm{d}y_*, \tag{4}$$

$$y \in (c,d), \quad t > 0,$$

the simplest external Dirichlet boundary conditions

$$\begin{aligned} &u_1(a_1,y,t) = 0,\ \ y \in (c,d); \quad u_1(x,c,t) = u_1(x,d,t) = 0,\ \ x \in (a_1,b_1),\\ &u_2(b_2,y,t) = 0,\ \ y \in (c,d); \quad u_2(x,c,t) = u_2(x,d,t) = 0,\ \ x \in (a_2,b_2), \end{aligned} \tag{5}$$

and the initial conditions

$$u_1(x,y,0) = u_{10}(x,y),\ \ (x,y) \in \Omega_1; \quad u_2(x,y,0) = u_{20}(x,y),\ \ (x,y) \in \Omega_2. \tag{6}$$

In particular, for a special choice of α_i and β_i such initial-boundary value problem describes radiative heat transfer in a system of absolutely black bodies [1, 2].

Throughout the paper we assume that the input data satisfy the usual regularity and ellipticity conditions

$$p_i(x,y),\ q_i(x,y) \in L_\infty(\Omega_i), \quad r_i(x,y) \in L_p(\Omega_i), \quad p > 1, \quad i = 1,2, \tag{7}$$

$$0 < p_{i0} \le p_i(x,y), \quad 0 < q_{i0} \le q_i(x,y), \quad \text{a.e. in } \Omega_i, \quad i = 1,2 \tag{8}$$

and

$$\alpha_i \in L_\infty(c,d), \quad \beta_i \in L_\infty\left((c,d)\times(c,d)\right), \quad i = 1,2. \tag{9}$$

In real physical problems (see [2]) we also often have

$$\alpha_i > 0\,, \quad \beta_i > 0, \quad i = 1,2.$$

By C, c_j and C_j we denote positive constants, independent of the solution of the IBVP and the mesh-sizes. C can take different values in the different formulas.

The aim of the present paper is to construct efficient finite difference scheme for the numerical solution of IBVP (1) – (6) and to investigate their convergence.

The layout of the paper is as follows. In Section 2 we briefly expose the properties of IBVP (1) – (6) and give some a priori estimates for its weak solution. In Section 3 we introduce meshes, finite-difference operators and discrete Sobolev-like normd. In Section 4 we define implicit finite difference scheme (FDS) approximating IBVP (1) – (6) and investigate its properties. A convergence rate estimate, compatible with the smoothness of the input data (up to a logarithmic factor of mesh-size), is obtained. In Section 5 we define factorized FDS and investigate its properties and convergence.

2. Weak solutions and a priori estimates

We introduce the product space

$$L = L_2(\Omega_1) \times L_2(\Omega_2) = \{v = (v_1,v_2)\,|\,v_i \in L_2(\Omega_i)\},$$

endowed with the inner product and the associated norm

$$(u,v)_L = (u_1,v_1)_{L_2(\Omega_1)} + (u_2,v_2)_{L_2(\Omega_2)}, \quad \|v\|_L = (v,v)_L^{1/2},$$

where

$$(u_i,v_i)_{L_2(\Omega_i)} = \int_{\Omega_i} u_i v_i\,\mathrm{d}x\,\mathrm{d}y, \quad i = 1,2.$$