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CELL CENTERED FINITE VOLUME METHODS USING TAYLOR SERIES EXPANSION SCHEME WITHOUT FICTITIOUS DOMAINS

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Abstract. The goal of this article is to study the stability and the convergence of cell-centered finite volumes (FV) in a domain $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ with non-uniform rectangular control volumes. The discrete FV derivatives are obtained using the Taylor Series Expansion Scheme (TSES), (see [4] and [10]), which is valid for any quadrilateral mesh. Instead of using compactness arguments, the convergence of the FV method is obtained by comparing the FV method to the associated finite differences (FD) scheme. As an application, using the FV discretizations, convergence results are proved for elliptic equations with Dirichlet boundary condition.

Key Words. finite volume methods, finite difference methods, Taylor series expansion scheme (TSES), convergence and stability, elliptic equations.

1. Introduction.

Finite volumes (FV) are widely used both in Engineering (see e.g. [4], [10] and [13]) and in Geophysical Fluid Dynamics (GFD) (see e.g. [11], [1] and [8]), because of their local conservation property on each control volume. From the mathematical and numerical analysis points of view, these methods are well studied for their stability and convergence, using a variety of methods to compute the fluxes (see e.g. [5], [6], [7], [9] and [14]). On a control volume in $\mathbb{R}^2$, one simple way to compute the flux along a boundary is to start with the difference of the given data at two cell centers divided by the length of the vector connecting those cell centers and then, taking the flux as the product of that quantity and the length of the boundary, which is the analog of the one dimensional case (see [5], [6], [7] and [9]). However this is not the best choice when the unit normal on the boundary is not parallel to the vector connecting the two cell centers; to deal with complicated meshes in $\mathbb{R}^2$, more efficient ways to compute the fluxes are needed. In this article, we consider the cell centered FV by Taylor Series Expansion Scheme (TSES), which permits to compute the fluxes on a general quadrilateral mesh in $\mathbb{R}^2$ (see [4] and [10]), and apply them to quasi- (but, non-) uniform meshes on Ω; we also intend to consider more general meshes in the future. For the mathematical analysis of the FV method, one specific difficulty is due to the "weak consistency" of FV. Indeed the companion discrete FV derivative arising in the discrete integration by parts does not usually converge strongly to the corresponding derivative of the limit function (see e.g. [6] or [9]). To overcome this difficulty, discrete compactness arguments have been used

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as in e.g. [6]. But here instead we consider the finite differences (FD) associated with the FV and compare the FV and FD spaces by defining a map between them. The convergence of the FV method is then inferred.

Our work is organized as follows. In Section 2, we describe the cell centered FV setting by TSES without using fictitious domains, but using instead "flat" domains at the boundary. In Section 3, we introduce an external approximation of $H^1_0(\Omega)$ using FV spaces V_h (see [3] and [15]), and show that the truncation error between a function in $H^1_0(\Omega)$ and its projection onto the FV space V_h tends to zero as the mesh sizes decrease. Due to the weak consistency of the FV, we are not able at this point to show that the external approximation of $H^1_0(\Omega)$ by the FV spaces is convergent. Instead, in Section 4, we present the FD method associated with this FV method and prove the stability and convergence of the external approximation of $H^1_0(\Omega)$ by the FD spaces $\widetilde{V}_h$ in Section 5. In Section 6, comparing the FV and FD spaces and thanks to the convergence of the FD, we obtain the convergence of the FV in the end. Finally, in Section 7, as an application, we demonstrate how one can use the FV method to approximate the solution of some typical elliptic equations with Dirichlet boundary condition, and, using our results, show the convergence of such an approximation via finite volumes to the solution of the original problem.

2. The Finite Volume Setting.

The domain is $\Omega = (0,1)\times(0,1)$ in $\mathbb{R}^2$. We set $x_0 = x_{\frac{1}{2}} = 0$, $x_{M+\frac{1}{2}} = x_{M+1} = 1$, $y_0 = y_{\frac{1}{2}} = 0$, $y_{N+\frac{1}{2}} = y_{N+1} = 1$ and we choose the nodal points $x_{i+\frac{1}{2}}$, $y_{j+\frac{1}{2}}$ for $1 \le i \le M-1$, $1 \le j \le N-1$,

$$\begin{aligned} 0 = (x_0 =) x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{M+\frac{1}{2}} (= x_{M+1}) = 1, \\ 0 = (y_0 =) y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N+\frac{1}{2}} (= y_{N+1}) = 1. \end{aligned} \tag{2.1}$$

We define the control volumes on Ω which appear on Fig. 1,

$$K_{i,j} = \begin{cases} (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}), & 1 \le i \le M, \ \ 1 \le j \le N, \\ (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times \{y_j\}, & 1 \le i \le M, \ \ j = 0, N+1, \\ \{x_i\} \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}), & i = 0, M+1, \ \ 1 \le j \le N. \end{cases} \tag{2.2}$$

Here, we have chosen flat control volumes at the boundary to handle and enforce the boundary conditions.

For $1 \le i \le M$, $1 \le j \le N$, the center of $K_{i,j}$ is

$$(x_i, y_j) = \Big(\frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}, \frac{y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}}{2} \Big). \tag{2.3}$$

We set

$$\begin{aligned} h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad k_j = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}, \quad 1 \le i \le M, \quad 1 \le j \le N, \\ h_{i+\frac{1}{2}} = x_{i+1} - x_i, \quad k_{j+\frac{1}{2}} = y_{j+1} - y_j, \quad 0 \le i \le M, \quad 0 \le j \le N, \end{aligned} \tag{2.4}$$

and, for convenience, we also set

$$h_0 = h_{M+1} = k_0 = k_{N+1} = 0. \tag{2.5}$$

Then we infer from (2.3)-(2.5) that

$$h_{i+\frac{1}{2}} = \frac{1}{2}\left(h_i + h_{i+1}\right), \ k_{j+\frac{1}{2}} = \frac{1}{2}\left(k_j + k_{j+1}\right), \ 0 \le i \le M, \ 0 \le j \le N, \tag{2.6}$$

and write the nodal points $x_{i+\frac{1}{2}}$, $y_{j+\frac{1}{2}}$ as proper weighted averages of the points x_i, x_{i+1}, y_j and y_{j+1} (see Fig. 2):