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ERROR ESTIMATES FOR AN OPTIMAL CONTROL PROBLEM GOVERNED BY THE HEAT EQUATION WITH STATE AND CONTROL CONSTRAINTS

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Abstract. In this work, we study priori error estimates for the numerical approximation of an optimal control problem governed by the heat equation with certain control constraint and ending point state constraint. By making use of the classical space-time discretization scheme, namely, finite element method with the space variable and backward Euler discretization for the time variable, we first project the original optimal control problem into a semi-discrete control and state constrained optimal control problem governed by an ordinary differential equation, and then project the aforementioned semi-discrete problem into a fully discrete optimization problem with constraints. With the help of Pontryagin's maximum principle, we obtain, under a certain reasonable condition of Slater style, not only an error estimate between the optimal controls for the original problem and the semi-discrete problem, but also an error estimate between the solutions of the semi-discrete problem and the fully discrete problem, which leads to an error estimate between the solutions of the original problem and the fully discrete problem. By making use of the aforementioned result, we also establish an numerical approximation for the exactly null controllability of the internally controlled heat equation.

Key Words. Finite element approximation, optimal control problem, the heat equation, ending point state constraint, error estimate.

1. Introduction

In this paper, we shall study error analysis for the discretization of an optimal control problem governed by the heat equation with certain control constraint and ending point (in time) state constraint, which will be introduced as follows. Let Ω be a bounded domain in $\mathbb{R}^d (d \leq 3)$ with a smooth boundary $\partial\Omega$, ω be an open subset of Ω and T be a positive number. We denote by Q the product set $\Omega \times (0,T)$ and by χ_ω the characteristic function of the subset ω. Write

$$\mathcal{K} = \{v \in L^2(0,T;L^2(\Omega)) \ ; \ \|v(t)\| \leq 1, \text{ for a.e. } t \in [0,T]\}$$

and

$$K = \{w \in L^2(\Omega) \ ; \ \|w\| \leq 1\}.$$

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Here and in what follows, we shall use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the usual norm and the inner product of the space $L^2(\Omega)$. The optimal control problem which we shall study reads

$$(P) \qquad \min_{u\in\mathcal{K}} \left\{ \frac{1}{2}\int_0^T \int_\Omega (y-y_d)^2 dxdt + \frac{1}{2}\int_0^T \int_\Omega u^2 dxdt \right\}$$

subject to

$$\left\{ \begin{array}{ll} \partial_t y - \triangle y = \chi_\omega u & \text{in } \Omega\times(0,T), \\ y = 0 & \text{on } \partial\Omega\times(0,T), \\ y(0) = y_0 & \text{in } \Omega, \end{array} \right. \tag{1.1}$$

and the ending point state constraint

$$y(T) \in K.$$

Here, $y_d \in L^2(Q)$ is a given target function and $y_0 \in H_0^1(\Omega)$ is a given initial data. Throughout the paper, the notation $y(t)$ stands for the value of the function $y : [0,T] \to L^2(\Omega)$ at the time t. As a matter of convenience, we shall often omit the notation t in functions of t and the notation (x,t) in functions of (x,t) whenever no confusion is possible.

We are going first set up a semi-discrete optimal control problem (P_h) projected by the original problem (P) in the sense of finite element, which is an optimal control problem governed by a system of linear ordinary differential equations with the ending point state constraint and a certain control constraint, and then establish a fully discrete optimal control problem $(P_{h\tau})$ projected by the aforementioned semi-discrete problem according to the classical backward Euler discretization scheme for the time variable. The problem $(P_{h\tau})$ is an optimal control problem governed by a system of linear algebraic equations with certain state and control constraints, and can be viewed as a problem of minimization of a quadratic function with convex constraints in a finite dimensional space, which, we assume and believe, can be solved numerically.

The purpose of this work is to obtain a convergence order for $L^2(Q)-$error between the optimal control for the original problem (P) and the solution of the fully discrete problem $(P_{h\tau})$. There should be several ways to reach such an aim. We shall make use of Pontryagin's maximum principle of the original problem (P), the semi-discrete problem (P_h) and the fully discrete problem (P_h) to establish first an error estimate between the solutions of the problems (P) and (P_h), and then an error estimate between the solutions for the problems (P_h) and $(P_{h\tau})$. The Pontryagin maximum principle of the problem (P) (also for the problems (P_h) and (P_h)) consists in a state equation, an adjoint equation, a transversality condition and a connection between the optimal control and the adjoint state, namely, the solution of the adjoint equation, through a variational inequality. The advantage that can be taken from the Pontryagin maximum principle in dealing with such error estimates is that one can expect quantitative expressions of the optimal controls via the adjoint states. Such relationships are helpful for us to get the desired error estimates in many cases.

However, due to the involvement of ending point state constraint, there will be a pair of multipliers in the space $\mathbb{R}\times L^2(\Omega)$ and appeared in the Pontryagin maximum principle for each problem among the problems (P), (P_h) and $(P_{h\tau})$. We denote them by (λ,μ), (λ_h,μ_h) and $(\lambda_{h\tau},\mu_{h\tau})$ for the problems (P), (P_h) and $(P_{h\tau})$, respectively. The multipliers λ, λ_h and $\lambda_{h\tau}$ appear in both variational inequalities and adjoint state equations, while the multipliers μ, μ_h and $\mu_{h\tau}$ arise in the initial