INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 2, Pages 281–302

A NEW FINITE VOLUME METHOD FOR THE STOKES PROBLEMS

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Abstract. A new finite volume method for solving the Stokes equations is developed in this paper. The finite volume method makes use of the BDM_1 mixed element in approximating the velocity unknown, and consequently, the finite volume solution features a full satisfaction of the divergence-free constraint as required for the exact solution. Optimal-order error estimates are established for the corresponding finite volume solutions in various Sobolev norms. Some preliminary numerical experiments are conducted and presented in the paper. In particular, a post-processing procedure was numerically investigated for the pressure approximation. The result shows a superconvergence for a local averaging post-processing method.

Key Words. finite volume methods, Stokes problems, discontinuous Galerkin method

1. Introduction

In scientific computing for science and engineering problems, finite volume methods are widely used and appreciated by users due to their local conservative properties for quantities which are of practical interest (e.g., mass or energy). Among many references, we would like to cite some which addresses theoretical issues such as stability and convergence [5, 6, 10, 11, 15, 16, 20, 21, 22, 8, 9, 10, 28, 29]. The goal of this paper is to investigate a finite volume method for the Stokes equations by using the well-known BDM elements [3] originally designed for solving second order elliptic problems. We intend to demonstrate how the BDM element can be employed in constructing finite volume methods for the model Stokes equations. The idea to be presented in the paper can be extended to problems of Stokes and Navier-Stokes type without any difficulty.

Mass conservation is a property that numerical schemes should sustain in computational fluid dynamics. This property is often characterized as an incompressibility constraint in the modeling equations. To sustain the mass conservation property for the Stokes equations, several finite element schemes have been developed to generate locally divergence-free solutions [12, 23]. In particular, a recent approach by using $H(\mathrm{div})$ conforming finite elements has been proposed and studied for a numerical approximation of incompressible fluid flow problems [13, 25, 26]. The main

1991 *Mathematics Subject Classification.* Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50 .

The research of Junping Wang was supported by the NSF IR/D program, while working at the Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

The research of Xiu Ye was supported in part by National Science Foundation Grant DMS-0813571.

advantage of using $H(\text{div})$ conforming elements is that the discrete velocity field is exactly divergence-free. Another advantage of using $H(\text{div})$ conforming elements is that the resulting linear or nonlinear algebraic systems can be easily decoupled between the velocity and the pressure unknowns, largely due to the availability of a computationally feasible divergence-free subspace for the velocity field. The purpose of this paper is to further explore the $H(\text{div})$ conforming elements in a finite volume context.

Our model Stokes equations are defined on a two-dimensional domain Ω. The standard Dirichlet boundary condition is imposed on the velocity field. The Stokes problem seeks a velocity $\mathbf{u}$ and a pressure p such that

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{3}$$

where the symbols Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively. $\mathbf{f}$ is the external volumetric force, and $\mathbf{g}$ is the velocity field on the boundary. For simplicity, we shall assume $\mathbf{g} = 0$ in the algorithmic description of the finite volume method. But the numerical experiments of Section 6 will be conducted for non-homogeneous data.

This paper is organized as follows. In Section 2, we introduce some notations that help us to give a technical presentation. In Section 3, a weak formulation is presented for the Stokes problem. Section 4 is dedicated to a presentation of a finite volume scheme by using the BDM element. In Section 5, we provide a theoretical justification for the finite volume scheme by establishing some error estimates in various norms. In addition to the standard H^1 and L^2 error estimates, we shall include an estimate for the pressure error in a negative norm, which ensures a certain superconvergence for the pressure when appropriate postprocessing methods are applied. In Section 6, a divergence-free finite volume formulation is discussed. Finally in Section 7, we present some numerical results that demonstrate the efficiency and accuracy of the new scheme.

2. Preliminaries and notations

We use standard notations for the Sobolev spaces $H^s(K)$ and their associated inner products $(\cdot,\cdot)_{s,K}$, norms $\|\cdot\|_{s,K}$, and semi-norms $|\cdot|_{s,K}$, $s \geq 0$ on a domain K. The space $H^0(K)$ coincides with $L^2(K)$, in which case the norm and inner product are denoted by $\|\cdot\|_K$ and $(\cdot,\cdot)_K$, respectively. The subscript K is suppressed when $K = \Omega$. Denote by $L_0^2(\Omega)$ the subspace of $L^2(\Omega)$ consisting of functions with mean value zero. Let $H(div,\Omega)$ be the space of all vector functions in $(L^2(\Omega))^2$ whose divergence is also in $L^2(\Omega)$, and $H_0(div,\Omega)$ be the space of all functions $\mathbf{v} \in H(div,\Omega)$ such that $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the unit outward normal vector.

Throughout the paper, we adopt the convention that a bold character in lower case stands for a vector. For simplicity, the Stokes problem (1)–(3) is assumed to have a full regularity of $\mathbf{u} \in (H^2(\Omega))^2$ and $p \in H^1(\Omega)$. In addition, we use $\lesssim$ ($\gtrsim$) to denote less than (greater than) or equal to up to a constant independence of the mesh size or other variables appeared in the inequality.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with characteristic mesh size h. Denote $\mathcal{E}_h$ to be the set of all edges in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h\backslash\partial\Omega$ to be the set of all interior edges. Each triangle $T \in \mathcal{T}_h$ is further divided into three subtriangles by connecting the barycenter C to the vertices A_k, $k = 1, 2, 3$, as shown in Figure 1.