INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 2, Pages 356–372

NONLINEAR MODEL REDUCTION USING GROUP PROPER ORTHOGONAL DECOMPOSITION

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(Communicated by Max D Gunzburger)

Abstract. We propose a new method to reduce the cost of computing nonlinear terms in projection based reduced order models with global basis functions. We develop this method by extending ideas from the group finite element (GFE) method to proper orthogonal decomposition (POD) and call it the group POD method. Here, a scalar two-dimensional Burgers' equation is used as a model problem for the group POD method. Numerical results show that group POD models of Burgers' equation are as accurate and are computationally more efficient than standard POD models of Burgers' equation.

Key words. model reduction, proper orthogonal decomposition, group finite element, nonlinear

1. Introduction

A challenge in the simulation of systems modeled by partial differential equations (PDE) is to reduce computational cost while preserving accuracy. To this end, much research in numerous aspects of the simulation of PDE has been performed. These efforts include attempts to reduce computational cost by improving algorithmic efficiency, developing parallel computing schemes, and applying model order reduction techniques. For example, reduced order modeling for fluid flows has seen extensive application of the Galerkin projection with proper orthogonal decomposition (POD) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

In this work, we submit a new method to reduce the cost of computing nonlinear terms in projection based reduced order models with global basis functions by extending ideas from the group finite element (GFE) method to POD[1]. We shall further refer to this approach as the group proper orthogonal decomposition (POD) method.

The GFE method, also known as product approximation, expresses the nonlinear terms of a PDE in grouped form - as the product of separate space and time dependent quantities. This leads to the spatial discretization of nonlinear terms being computed *once* before integration and a substantial reduction in computational cost [12, 13, 14]. Here, instead of projecting grouped nonlinear terms onto a local finite element basis, we show that the projection of grouped nonlinear terms onto a set of global basis functions reduces the cost of simulation due to symmetry in the nonlinear terms. Although a Galerkin projection onto a POD basis is used here for illustration, we anticipate this method to be generally applicable to other global basis functions and Petrov-Galerkin projections.

Received by the editors September 5, 2008 and, in revised form, August 3, 2009.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.
This research was supported by Air Force Office of Scientific Research grant FA9550-07-1-0540. The authors would also like to thank Belinda Batten for her helpful comments and support.

[1] We note that this method has also been independently investigated by Max Gunzburger (private communication).

To determine the accuracy of the group POD method, computational solutions of group POD and standard POD models of Burgers' equation are compared to analytical manufactured benchmark solutions [15, 16, 17, 18, 19]. Our experiments show close agreement between simulations of the group POD and standard POD models of Burgers' equation.

To assess the computational cost of the group POD method, total integration times and operation counts for the nonlinear terms of the group POD model of Burgers' equation are compared to the standard POD model. For the quadratic nonlinearity of Burgers' equation, our results show the group POD method provides a clear computational advantage over the standard POD approach in terms of operation count and total integration time.

Following this introduction, we provide background on POD and the GFE method. The standard and group POD models of Burgers' equation are developed in Section 3, followed by their implementation and operation counts in Section 4. Section 5 contains numerical results which demonstrates that group POD models of Burgers' equation are as accurate and are more efficient than the standard POD form. Finally, we provide a mathematical extension of the group POD method to cubic nonlinearities in Section 6.

2. Background

We begin by providing background on two concepts key to the group POD method: proper orthogonal decomposition (POD) itself and the group finite element (GFE) method. While POD offers computational advantages through a reduction in order, the GFE method offers computational gains through the construction of nonlinear terms. Following the background on POD, we illustrate the computational advantage of grouping the nonlinear terms with the GFE form of Burgers' equation.

2.1. Notation. To describe POD based model reduction for partial differential equations, we use the following notation. Let X be a Hilbert space with its inner product and corresponding norm denoted $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively. A function, u, is in $L^2(0,T;X)$ if for each $0 \leq t \leq T$, $u(t)$ is in X, and $\int_0^T \|u(t)\|_X^2 \, dt < \infty$.

2.2. Proper Orthogonal Decomposition (POD). At the turn of the twentieth century, the closest fitting lines or planes to a set of points in space was investigated by Pearson [20]. Independently, almost three decades later, a similar treatment appeared by Hotelling where the "method of principal components" was coined [21]. The analysis presented in [20] and [21] formed the linear algebraic approach to what many now call proper orthogonal decomposition (POD).

Since the work of Pearson and Hotelling, many have studied or used POD in a range of fields such as oceanography [22], fluid mechanics [1, 2, 4], system feedback control [23, 24, 25, 26, 27, 28], and system modeling [5, 8, 10, 29]. Following many predecessors, we use POD as tool for model reduction where the POD of a function, $u \in L^2(0,\infty;X)$, gives a basis that best represents $u \in L^2(0,\infty;X)$ in a mean-square sense [8, 9].

The method of snapshots is a practical approach to compute the POD of a function known pointwise in time. The method of snapshots may be derived from the continuous POD operator by assuming a piecewise constant representation of u in time as shown in [9]. The remainder of this section outlines the procedure.