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SHISHKIN MESHES IN THE NUMERICAL SOLUTION OF SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

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Dedicated to G. I. Shishkin on the occasion of his 70th birthday

Abstract. This article reviews some of the salient features of the piecewise-uniform Shishkin mesh. The central analytical techniques involved in the associated numerical analysis are explained via a particular class of singularly perturbed differential equations. A detailed discussion of the Shishkin solution decomposition is included. The generality of the numerical approach introduced by Shishkin is highlighted. The impact of Shishkin's ideas on the field of singularly perturbed differential is assessed in this selective review of his research output over the past thirty years.

Key Words. singularly perturbed problems, Shishkin mesh, Shishkin solution decomposition.

1. Introduction

This review paper addresses the numerical solution of computationally challenging singularly perturbed differential equations and, in particular, how this area of numerical analysis was enhanced by the contributions of the Russian mathematician Grigorii Ivanovich Shishkin. This review is not comprehensive in the sense that no attempt is made to give an overview of all current activities in the area, not even an overview of all the contributions made by Shishkin over the years (as this topic would warrant a monograph by itself). We aim to give a simple and self-contained presentation of those techniques by Shishkin that, in our opinion, have had most impact on the area. In particular, we shall describe the construction of Shishkin meshes and the Shishkin solution decomposition. We also aim to highlight the generality of Shishkin's approach, which is evidenced by a broad range of problems to which Shishkin has applied his methodology. Finally, we shall review some of the literature to demonstrate how Shishkin's ideas were employed and, furthermore, blended with other techniques by authors other than Shishkin.

Singularly perturbed differential equations are typically characterized by a small parameter ε multiplying some or all of the highest order terms in the differential equation. In general, the solutions of such equations exhibit multiscale phenomena. Within certain thin subregions of the domain, the scale of some partial derivatives is significantly larger than other derivatives. We call these thin regions of rapid change, boundary or interior layers, as appropriate. For small values of ε, an analytical approximation to the exact solution can be generated using the techniques of matched asymptotic expansions [38, 66, 100, 103]. Such asymptotic approximations identify the fundamental nature of the solution across the different scales.

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Classical computational approaches to singulary perturbed problems are known to be inadequate as they require extremely large numbers of mesh points to produce satisfactory computed solutions [26, 73]. Throughout the paper, we focus on robust numerical methods, also called uniformly convergent or parameter-uniform methods, that converge in the discrete maximum norm independently of the size of the singular perturbation parameter(s).

Shishkin has been producing publications on singularly perturbed problems since 1974; see [25]. Motivated by physical problems, he has developed over the intervening years a distinctive approach to both constructing and analysing appropriate parameter-uniform numerical methods for singularly perturbed problems. In general, he combines inverse-monotone finite difference schemes with layer-adapted piecewise-uniform meshes (often called Shishkin meshes by others) to produce a computed solution (with accuracy measured in the discrete maximum norm), whose global convergence is guaranteed independently of any singular perturbation parameter present in the problem.

Throughout his publications, Shishkin seeks out simplicity in the design of computational algorithms for singularly perturbed problems; in particular, simplicity in the design of the mesh. In §2, using a constant-coefficient ordinary differential equation, we identify the deficiencies associated with a uniform mesh. We then retrace the path followed by Shishkin and derive the necessary conditions for a piecewise-uniform mesh to support a uniformly convergent method. We also briefly discuss generalizations of Shishkin mesh in more than one dimension. In the context of simplicity, note that a piecewise-uniform mesh only differs from a uniform mesh at one or a few transition points.

A useful analytical technique developed by Shishkin, which is used in the numerical analysis associated with these layer-adapted meshes, is a particular solution decomposition. Note that this decomposition is not an asymptotic expansion, as there is no remainder term present. The decomposition technique involves defining some associated problems on extended domains in order to minimize the imposition of additional compatibility conditions and then employing Schauder-type estimates to determine bounds on the derivatives of each component in the decomposition. These a priori estimates on the derivatives of the exact solution are used both to identify rate constants in the layer functions (which are utilized in the mesh design) and in the error analysis [69, 70, 82, 94]. In §3, we first recall a solution decomposition by Bakhvalov, whose approach contains some of the key aspects of the Shishkin decomposition, and then describe a Shishkin decomposition for a linear convection–diffusion problem in two space dimensions.

In our opinion, a significant attribute of the Shishkin mesh is the fact that, once the location and width of all possible layers are identified, the same methodology is applicable to various different classes of singularly perturbed problems. In §4, we describe some problem classes for which a Shishkin mesh has been constructed and Shishkin's analysis technique has been applied, to emphasize the extent and generality of the Shishkin approach.

In the final section, we review the recent literature related to Shishkin's publications. We exclude papers authored or co-authored by Shishkin himself from this final section. We note that the increasing number of papers that involve a Shishkin mesh and/or a Shishkin decomposition is a clear indication of the significant impact that Shishkin's research has had on the area of numerical methods for singularly perturbed differential equations.