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POINTWISE APPROXIMATION OF CORNER SINGULARITIES FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS WITH CHARACTERISTIC LAYERS

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This paper is dedicated to 70th anniversary of Grigorii I. Shishkin

Abstract. A Dirichlet problem for a singularly perturbed steady-state convection-diffusion equation with constant coefficients on the unit square is considered. In the equation under consideration the convection term is represented by only a single derivative with respect to one coordinate axis. This problem is discretized by the classical five-point upwind difference scheme on a rectangular piecewise uniform mesh that is refined in the neighborhood of the regular and the characteristic boundary layers. It is proved that, for sufficiently smooth right-hand side of the equation and the restrictions of the continuous boundary function to the sides of the square, without additional compatibility conditions at the corners, the error of the discrete solution is $O(N^{-1}\ln^2 N)$ uniformly with respect to the small parameter, in the discrete maximum norm, where N is the number of mesh points in each coordinate direction.

Key Words. parabolic boundary layers, elliptic equation, piecewise uniform mesh, corner singularities.

1. Introduction

In the unit square $\Omega = (0,1)^2$ with the boundary $\partial\Omega$ the following problem is considered

$$Lu := -\varepsilon\Delta u + a\frac{\partial u}{\partial x} + qu = f(x,y), \quad (x,y) \in \Omega, \quad u\Big|_{\partial\Omega} = g, \tag{1}$$

where

$$a = \text{const} > 0, \quad q = \text{const} > 0, \tag{2}$$

and $\varepsilon \in (0,1]$ is a small parameter.

Let Γ_k be the sides of the square Ω enumerated counter-clockwise, beginning with $\Gamma_1 = \{(x,y) \in \partial\Omega \mid x = 0\}$ and let $a_k = (x_k, y_k)$ be its vertices, enumerated in the same way with $a_1 = (0,0)$. Let also $g_k := g\Big|_{\Gamma_k}$ denote a restriction of the boundary function g to the side Γ_k of the square Ω.

Numerous investigations (see [11] and the references) show that the solution to problem (1)-(2) has a complicated structure, involving a regular boundary layer in the neighborhood of the right boundary Γ_3, two characteristic layers in the neighborhood of the bottom and the top boundaries Γ_2 and Γ_4, corner layers with corner singularities in the neighborhood of the vertices a_2 and a_3, and corner singularities in the neighborhood of the inflow vertices a_1 and a_4; see Figure 1.

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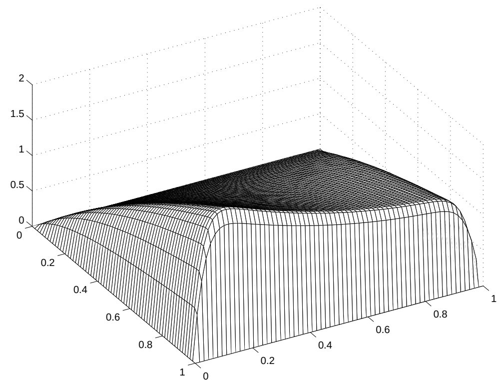

FIGURE 1. Solution of model problem (1), (30); $\varepsilon = 2^{-9}$.

In the recent work [5], which was improved in [6], a detailed analysis is given of the solution to problem (1)-(2) with ε-explicit estimates of all its derivatives in general case $g \notin C(\partial\Omega)$. A few years earlier, the solution of this problem, under assumptions that the compatibility conditions of the first order are satisfied, was analyzed in [10]. In [9] equation (1) with variable convection coefficient a was investigated, but under very severe compatibility conditions at the corner points excluding appearance of corner singularities both in the solution itself and in its derivatives up to desired order for equation (1) and for the reduced equation as well. Under the same assumptions, in [9] the convergence of classical five-point upwind difference scheme is analyzed for which, on a Shishkin mesh, the convergence estimate of $O(N^{-1}\ln^2 N)$ is obtained, where N is the number of mesh points in each coordinate direction. Earlier in [10], a comment was made that one might get the error estimate of $O(N^{-1}\ln N)$ for this scheme (using the obtained solution decomposition). In spite of heaviness of compatibility conditions, in most works dealing with analysis of numerical methods for singularly perturbed equations in a rectangle, such assumptions are made in order to provide smoothness to the solution being approximated. The book of Shishkin [12] is an exception. In this book for some problems certain compatibility conditions are posed, while those are not posed for other problems. But for the cases when the compatibility conditions are not posed, estimates, obtained in [12] (for a much more general problem than (1), (2)), give low orders of convergence. For example, for the finite-difference scheme (9) applied to problem (1), (2), only the error bound $O((N^{-1}\ln^2 N)^{1/14})$ is given in [12].

In recent years for some singularly perturbed problems the author of this paper has succeeded in carrying a more thorough analysis of the convergence rate of difference schemes, when the problem data at the corner points have minimal compatibility (only the continuity is required). Thus, in [1], [2] for a singularly perturbed reaction-diffusion equation on a unit square, with the Dirichlet and the Dirichlet-Neumann boundary conditions, the error of the classical difference scheme on a Shishkin mesh is proved to be $O(N^{-2}\ln^2 N)$. However, in the case of Dirichlet-Neumann boundary conditions, it was necessary to use an additional power refinement in the neighborhood of those corner points where boundary conditions of different types were imposed at the adjacent sides. A similar situation occurs in [4] too, where the reaction-diffusion equation in an L-shaped domain is investigated. In [3] the convection-diffusion equation in a rectangle with a regular boundary layer is considered. For this problem, singularities at different corner points are of very