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FINITE VOLUME APPROXIMATION OF TWO-DIMENSIONAL STIFF PROBLEMS

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This paper is dedicated to G.I. Shishkin on the occasion of his 70th birthday

Abstract. Continuing an earlier work in space dimension one, the aim of this article is to present, in space dimension two, a novel method to approximate stiff problems using a combination of (relatively easy) analytical methods and finite volume discretization. The stiffness is caused by a small parameter in the equation which introduces ordinary and corner boundary layers along the boundaries of a two-dimensional rectangle domain. Incorporating in the finite volume space the boundary layer correctors, which are explicitly found by analysis, the boundary layer singularities are absorbed and thus uniform meshes can be preferably used. Using the central difference scheme at the volume interfaces, the proposed scheme finally appears to be an efficient second-order accurate one.

Key Words. Finite volume methods, boundary layers, correctors, asymptotic analysis, singularly perturbed problems, stiff problems

1. Introduction

We consider convection-dominated problems in a two-dimensional domain:

$$\begin{cases} div(-\varepsilon \nabla u^\varepsilon - \mathbf{b} u^\varepsilon) = f \text{ in } \Omega, \\ u^\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, $div(\mathbf{b}) = 0$, $\mathbf{b} = (b_1, b_2)^T$ with $b_1, b_2 \geq \delta > 0$ and $\varepsilon > 0$, and $b_1 = b_1(x,y), b_2 = b_2(x,y)$ and $f = f(x,y)$ are sufficiently smooth. When ε is small, e.g. $0 < \varepsilon << \delta$, the solutions u^ε of Problem (1.1) possess boundary layers at the outflow boundaries, that is, $x = 0$, $y = 0$. For the analysis of boundary layers problems the reader is referred to e.g. [4], [5], [8], [9], [11], [21], [23], [25] and [27], and for the numerical approach to e.g. [24], [7], [12], [13], [15] - [19], [22] and [26]. Notice that the boundary $\partial\Omega$ of Ω is nowhere characteristic. Since $div(\mathbf{b}) = 0$, we also note that $div(-\varepsilon\nabla u - \mathbf{b}u) = -\varepsilon\Delta u - \mathbf{b}\cdot\nabla u$ and the well-posedness of Problem (1.1) in the Sobolev space $H_0^1(\Omega)$ is standard, thanks to the Lax-Milgram theorem. Furthermore, we can verify the following norm estimates for the solutions u^ε.

Lemma 1.1. *Let $f = f_1 + f_2 + f_3$. There exists a positive constant κ, independent of ε, such that*

$$\begin{cases} |u^\varepsilon|_{L^2(\Omega)} \leq \kappa N_\varepsilon(f), \\ |u^\varepsilon|_{H^1(\Omega)} \leq \kappa \varepsilon^{-\frac{1}{2}} N_\varepsilon(f), \\ |u^\varepsilon|_{H^2(\Omega)} \leq \kappa \varepsilon^{-\frac{3}{2}} N_\varepsilon(f), \end{cases} \tag{1.2}$$

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where $N_\varepsilon(f) = |f_1|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{2}}|x(L_1-x)f_2|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{2}}|y(L_2-y)f_3|_{L^2(\Omega)}$.

Proof. To estimate $u^\varepsilon = u$, we write $u = e^{-x}v$ and then we have

$$\begin{cases} -\varepsilon\Delta v - div(\mathbf{b}v) + (b_1-\varepsilon)v + 2\varepsilon v_x = e^x f \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

We first observe that

$$-\int_\Omega div(\mathbf{b}v)v = -\int_\Omega (\mathbf{b}\cdot\nabla v)v = \int_\Omega div(\mathbf{b})\frac{v^2}{2} = 0. \tag{1.4}$$

Multiplying then $(1.3)_1$ by v and integrating over Ω, we find

$$\begin{aligned} \varepsilon|v|^2_{H^1} + (\delta-\varepsilon)|v|^2_{L^2} &\le \kappa|x(L_1-x)f_2|_{L^2}|(\frac{1}{x}+\frac{1}{L_1-x})v|_{L^2} \\ &+ \kappa|y(L_2-y)f_3|_{L^2}|(\frac{1}{y}+\frac{1}{L_2-y})v|_{L^2} + \kappa|f_1|_{L^2}|v|_{L^2} \\ &\le \kappa(|f_1|_{L^2}|v|_{L^2} + |x(L_1-x)f_2|_{L^2}|v|_{H^1} + |y(L_2-y)f_3|_{L^2}|v|_{H^1}). \end{aligned} \tag{1.5}$$

In (1.5) we have used the Hardy inequality (see e.g. [19], [10]) in the form:

$$\left|\frac{u}{x}\right|_{L^2(\Omega)} \le \kappa|u|_{H^1(\Omega)}, \text{ for } u = 0 \text{ at } x = 0. \tag{1.6}$$

The first two inequalities (1.2) follow promptly from (1.5). Then the H^2 regularity and the H^2 estimate immediately follow from (1.1). □

Convection-dominated problems appear in many applications where convection plays an important role, as for instance weather-forecasting, oceanography, transport of contaminant, etc. (see e.g. [3]). In this article we build a novel method to approximate, numerically, two-dimensional convection-dominated problems and via numerical examples the new numerical scheme is proved efficient and accurate.

Before we proceed we analyze below the stiffness of the solutions due to the small parameter ε.

2. Singular perturbation analysis

In general, functions like u^ε can be decomposed into a relatively slow (smooth) part u^s and a fast part u^f, i.e. $u^\varepsilon = u^s + u^f$. Using standard classical numerical methods the slow part u^s can be easily approximated, but the fast part u^f produces large approximation errors due to the stiff gradients. Introducing the so-called correctors which appear below we will resolve such issues for the problem under consideration. The singular perturbation analysis provides the two important settings. One is to locate the stiff parts, namely the boundary layers; we will modify them and construct appropriate forms of u^f which absorb the boundary layer singularities. The other is to impose the boundary conditions for the slow parts u^s which are close to the limit solutions.

Writing (1.1) in a non-divergence form we first construct the limit solution of u^ε in (1.1), i.e. when $\varepsilon = 0$. That is, we find

$$\mathbf{b}\cdot\nabla u^0 = f \text{ in } \Omega, \tag{2.1}$$

and then impose the zero boundary conditions at the inflows, i.e. $x = 1$ or $y = 1$. This choice of the boundary condition for (2.1) will be justified a posteriori by our convergence result. The existence and uniqueness of a solution $u^0 \in L^2(\Omega)$ of (2.1) satisfying the zero boundary conditions at the inflows is well-known. In what follows we will assume that u^0 is as regular as needed. Such regularity results may