INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 3, Pages 477–490

AN ENRICHED SUBSPACE FINITE ELEMENT METHOD FOR CONVECTION-DIFFUSION PROBLEMS

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This paper is dedicated to G.I. Shishkin on the occasion of his 70th birthday

Abstract. We consider a one-dimensional convection-diffusion boundary value problem, whose solution contains a boundary layer at the outflow boundary, and construct a finite element method for its approximation. The finite element space consists of piecewise polynomials on a uniform mesh but is enriched by a finite number of functions that represent the boundary layer behavior. We show that this method converges at the optimal rate, independently of the singular perturbation parameter, when the error is measured in the energy norm associated with the problem. Numerical results confirming the theory are also presented, which also suggest that in the case of variable coefficients, the number of enrichment functions need not be as high as the theory suggests.

Key Words. finite element method, boundary layers, enriched subspace.

1. Introduction

Let $p > 0$, $q > 0$ be smooth functions, let $\varepsilon \in (0, 1]$, and consider the problem

$$Lu := -\varepsilon u'' + p(x)u' + q(x)u = f(x) \text{ in } (0, 1),\ u(0) = u(1) = 0. \tag{1}$$

It is well-known that the solution to this problem has a boundary layer at $x = 1$, and that an accurate, robust numerical solution can be obtained by putting a highly refined mesh, often called the "Shishkin mesh", near this boundary point [7, 9] ; see also [8] for other mesh choices used in conjunction with the high order p and hp versions of the finite element method (FEM). In this paper we suggest an alternate way to obtain an accurate and robust numerical method. We use a FEM with a uniform mesh. The finite element subspace consists of the usual piecewise polynomials subspace, enriched by a finite number of functions that represent the boundary layer behavior. It is shown that this results in a numerical solution with an ε-uniform error bound in the energy norm associated with the problem. Numerical results are given to illustrate the method.

Perhaps the first use of boundary layer enrichment was given in the paper of Han and Kellogg, [2]. Subsequent work related to this paper is found in Cheng-Temam, [1], which also considers a singularly perturbed ordinary differential equation. The paper [1] is restricted to an equation with constant coefficients, and uses only piecewise linear functions plus an enrichment function. The results are analogous to those of the present paper. Jung and Temam [3, 4, 5] have applied enriched finite

Received by the editors April 5, 2009 and, in revised form, October 12, 2009.
2000 *Mathematics Subject Classification.* 65N30, 65N15, 65L60.

elements to a model singularly perturbed convection diffusion problem whose solution involves both ordinary and parabolic boundary layers. It would be interesting to apply the enriched technique to problems with interior layers.

Section 2 gives some properties of the solution to (1) that are needed for our error analysis. Section 3 formulates the enriched FEM and gives the error analysis. Section 4 presents some numerical results.

We require that the functions p, q, f are sufficiently smooth. Also we assume

$$0 < p_{\min} \leq p(x) \leq p_{\max} < \infty, \tag{2a}$$

$$q(x) > 0 \text{ in } [0,1], \tag{2b}$$

$$q(x) - \frac{1}{2}p'(x) > 0 \text{ in } [0,1]. \tag{2c}$$

We let $\|w\|_k$ denote the norm in the Sobolev space $H^k(0,1)$, and we use the notation

$$\|w\|_{k,\infty} = \sup\{|w^{(j)}(x)| : x \in [0,1],\ j = 0, \cdots, k\}.$$

We also use $D_x^j w$ as well as $w^{(j)}(x)$ to denote the j^{th} derivative of w with respect to x. When there is no confusion, we will omit the subscript/variable and simply write $D^j w$ or $w^{(j)}$. The letter C denotes a positive number that may be different in different instances, but is always independent of ε and the mesh spacing h.

2. Solution properties

The solution properties for the problem (1) are well-known and may be found, for example, in [7]. These properties are stated here in a form that is useful for our analysis.

From (2b), solutions of (1) satisfy the maximum principle. Therefore the problem (1) has a solution u, and $\|u\|_{0,\infty} \leq C\|f\|_{0,\infty}$. For derivative bounds we cite [7, Lemma 1.8]:

$$|u^{(k)}(x)| \leq C(f)\big(1 + \varepsilon^{-k} e^{-p_{\min}(1-x)/\varepsilon}\big).$$

Examining the proof one obtains

$$|u^{(k)}(x)| \leq C\|f\|_{k,\infty}\big(1 + \varepsilon^{-k} e^{-p_{\min}(1-x)/\varepsilon}\big). \tag{3}$$

We now give a formal asymptotic expansion of the solution. This expansion is also given in [7, p.22], but we derive it in greater detail in order to obtain the information contained in Lemma 1.

Let $V_{n-1}(x) = \sum_{j=0}^{n-1} \varepsilon^j v_j(x)$. Then

$$\begin{aligned} LV_{n-1} &= \sum_{j=0}^{n-1} \big[-\varepsilon^{j+1} v_j'' + p\varepsilon^j v_j' + q\varepsilon^j v_j \big] \\ &= pv_0' + qv_0 + \sum_{j=1}^{n-1} \varepsilon^j \big[pv_j' + qv_j - v_{j-1}''\big] - \varepsilon^n v_{n-1}''. \end{aligned} \tag{4}$$

Define the functions v_j by

$$\begin{aligned} &pv_0' + qv_0 = f,\ v_0(0) = 0, \\ &pv_j' + qv_j = v_{j-1}'',\ v_j(0) = 0 \text{ for } j = 1, \cdots, n-1. \end{aligned}$$