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A POSTERIORI ERROR ESTIMATION FOR A SINGULARLY PERTURBED PROBLEM WITH TWO SMALL PARAMETERS

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Abstract. A singularly perturbed two-point boundary-value problem of reaction-convection-diffusion type is considered. The problem involves two small parameters that give rise to two boundary layers of different widths. The problem is solved using a streamline-diffusion FEM (SDFEM).
A robust *a posteriori* error estimate in the maximum norm is derived. It provides computable and guaranteed upper bounds for the discretisation error. Numerical examples are given that illustrate the theoretical findings and verify the efficiency of the error estimator on *a priori* adapted meshes and in an adaptive mesh movement algorithm.

Key Words. reaction-convection-diffusion problems, finite element methods, a posteriori error estimation, singular perturbation

1. Introduction

Consider the reaction-convection-diffusion problem of finding $u \in C^2(0,1) \cap C[0,1]$ such that

$$\mathcal{L}u := -\varepsilon_d u'' - \varepsilon_c b u' + cu = f \quad \text{in} \quad (0,1) \quad \text{and} \quad u(0) = u(1) = 0, \tag{1}$$

where $\varepsilon_d \in (0,1]$ and $\varepsilon_c \in [0,1]$ are small parameters, while $b \in C^1(0,1)$ and $c, f \in C(0,1)$ are assumed to satisfy

$$b \geq 1, \quad c \geq 1 \quad \text{and} \quad \varepsilon_c b' + c \geq 0 \quad \text{in} \quad (0,1). \tag{2}$$

The positivity of b and c is essential, while the third inequality merely provides a maximal threshold value for ε_c for which the analysis in the paper is valid.

The standard weak formulation of (1) is: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) := \varepsilon_d(u',v') - \varepsilon_c(bu',v) + (cu,v) = (f,v) =: f(v) \quad \forall\, v \in H_0^1(0,1). \tag{1'}$$

The solution of (1) typically exhibits two boundary layers of different widths at the two endpoints of the domain. Because of the presence of these layers standard numerical methods fail to give accurate approximations. Unless a prohibitively large number of mesh points is used, the layers are not resolved, and the rate of convergence achieved by the method is far less than that obtain in the non-singularly perturbed case.

The goal is to construct so-called *robust* or *uniformly convergent* methods. This means that for a fixed number of mesh points, the accuracy and rate of convergence is guaranteed, irrespective of the magnitude of the perturbation parameters. Approaches for achieving this aim include the use of meshes that contain a concentration of points in the region of the boundary layers. The piecewise uniform meshes of Shishkin [18], and the graded meshes of Bakhvalov [2] are examples of

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such. The construction of these meshes depends strongly on *a priori* information of the solution and its derivatives.

Adapted numerical methods for (1) were first analysed by Shishkin and Titov [22]. They consider an exponentially fitted finite difference scheme on a uniform mesh. This method is shown to be convergent, uniformly in the parameters ε_d and ε_c, in the discrete maximum norm. The order of convergence is at least $N^{-2/5}$, where N is the number of mesh intervals.

About 25 years after the work by Shishkin and Titov a number of authors started to investigate standard numerical methods on special layer-adapted meshes. Linß and Roos [17] studied a first-order upwinded difference scheme on a piecewise uniform Shishkin mesh. Uniform convergence of $\mathcal{O}\left(N^{-1}\ln N\right)$ was established. A theory for this method on general meshes was developed in [14].

Second-order upwind schemes were considered by Roos and Uzelac [21] (using a SDFEM approach) and by Gracia et all. [8]. Both papers establish uniform convergence of $\mathcal{O}\left(N^{-2}\ln^2 N\right)$ on Shishkin meshes.

While these *a priori* results establish the asymptotic behaviour of the error as the mesh is refined, it cannot give guaranteed upper bounds for the error on a particular mesh. The constant in the error bound, though independent of the perturbation parameters, depends on the exact solution u which in turn is unknown.

The main contribution of the present study is in establishing *a posteriori* error bounds which provide upper bounds on the error of the SDFEM. These days, *a posteriori* error estimates for classical problems, i.e. problems that are not singularly perturbed, are well established, see for example the monographs [1] and [23]. Results are also available for the SDFEM applied to convection-diffusion problems [24]. All these analyses are set in an L_2- and energy-norm framework. However, for (1) these norms fail to capture the layers. Therefore, we are interested in *maximum-norm* error bounds.

For singularly perturbed problems, *a posteriori* error analyses in the maximum norm have been pioneered by Kopteva both for convection-diffusion problems in 1D [10] and for reaction-diffusion problems in 1-3D [11, 12, 7]. In the present paper, *a posteriori* error bounds for a single equation with two independently small parameters are derived for the first time. In a certain sense it generalises the 1D results by Kopteva for both reaction-diffusion ($\varepsilon_c = 0$) and convection-diffusion ($\varepsilon_c = 1$).

Outline. The paper is organised as follows. In § 2 we study properties of the continuous problem (1). In particular bounds for the Greens function associated with $\mathcal{L}$ are derived that are essential in the later error analysis. The SDFEM for ($1'$) is introduced in § 3, while § 4 is devoted to its *a posteriori* error analysis. An adaptive mesh movement algorithm is adapted from the literature in § 5. The article closes with results of some numerical experiments.

Notation. Throughout C denotes a generic positive constant that is independent of the parameters ε_d and ε_c and of N, the number of mesh points. We use $\|\cdot\|_D$ to denote the norm in $L_\infty(D)$. When $D = (0,1)$ we drop the D from the notation.

2. Properties of the continuous problem

The solution of (1) and its Green's function can be described by means of the two roots of the characteristic equation

$$-\varepsilon_d\lambda(x)^2 - \varepsilon_c b(x)\lambda(x) + c(x) = 0. \tag{3}$$