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THE TWO-LEVEL LOCAL PROJECTION STABILIZATION AS AN ENRICHED ONE-LEVEL APPROACH. A ONE-DIMENSIONAL STUDY

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This paper is dedicated to G.I. Shishkin on the occasion of his 70th birthday

Abstract. The two-level local projection stabilization is considered as a one-level approach in which the enrichments on each element are piecewise polynomial functions. The dimension of the enrichment space can be significantly reduced without losing the convergence order. For example, using continuous piecewise polynomials of degree $r \geq 1$, only one function per cell is needed as enrichment instead of r in the two-level approach. Moreover, in the constant coefficient case, we derive formulas for the user-chosen stabilization parameter which guarentee that the linear part of the solution becomes nodally exact.

Key Words. local projection stabilization, finite elements, Shishkin mesh, convection diffusion equation

1. Introduction

It is well-known that standard Galerkin finite element discretizations applied to convection-diffusion problems show spurious oscillations unless the mesh is adapted to the boundary layers of the solutions [21]. But even in the case of layer adapted meshes it makes sense to use stabilized finite element schemes in order to reduce sensitivities of the solutions on the choice of mesh parameters. Residual based stabilization methods like the streamline upwind Petrov-Galerkin (SUPG) stabilization, proposed in [5] and at first analyzed for a scalar convection-diffusion equation in [19], is a prominent example of stabilized schemes. They rely on adding weighted residuals to the standard Galerkin method to enhance stability without losing consistency.

Recently, local projection stabilization (LPS) [2, 3, 9, 10, 12, 13, 17, 18, 20] methods have become quite popular, in particular because of their commutation properties in optimization problems [4] and stabilization properties similar to those of the SUPG method [11]. In contrast, to residual based stabilization methods the LPS is no longer consistent. However, taking rich enough projection spaces any desired consistency order can be achieved. As shown in [17], the key issue in analyzing the error of LPS schemes is the existence of an interpolation for which the error is orthogonal to the projection space. It turns out, that a local inf-sup condition for the approximation and projection space is sufficient to modify an interpolation into the approximation space in such a way that the additional orthogonality property holds [17]. Two main approaches of LPS have been considered in the literature to fulfil the local inf-sup condition. In the one-level approach, a standard finite

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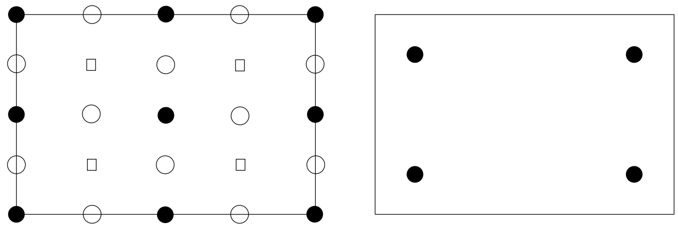

FIGURE 1. Degrees of freedom of two-level methods on one macro cell. Enriched biquadratic approximation spaces (left) and bilinear projection space (right).

element space is chosen as the projection space to guarantee the consistency order. Then, the approximation space is (if necessary) enriched such that the local inf-sup condition holds. In the two-level approach, a standard finite element space is chosen as the approximation space and the projection space is thinned out to a space on the next coarser mesh level to satisfy the local inf-sup condition.

The main objective of this paper is to show that the two-level variant of the LPS can be also considered as an enriched one-level method on the coarser mesh. This enables us to reduce the degrees of freedom in the two-level method without losing the convergence order. For example, on a rectangular coarse mesh 16 degrees of freedom (squares and non-filled circles) to the 9 degrees of freedom (filled circles) have to be added per macro cell to generate the full biquadratic approximation space on the next finer mesh level, see Figure 1. However, for satisfying the local inf-sup condition with respect to the associated 4-dimensional space of bilinear functions the 4 degrees of freedom indicated by squares are enough and lead to a reduced two-level method with optimal convergence order.

Here, we restrict our attention only to the one-dimensional case in which already one additional function per macro cell is sufficient. Furthermore, for constant coefficients we can chose the stabilization parameter in such a way such that the piecewise linear part of the LPS becomes nodally exact. Although such a strong result cannot be expected in the multi-dimensional case a considerable reduction of degrees of freedom in the two-level method without losing the convergence is still possible. We will address the case of higher dimensions in a forthcoming paper.

In the following, we use the standard notations for Sobolev spaces $H^k(D)$, $H_0^k(D)$, $L^2(D) = H^0(D)$ together with their norms and semi-norms $\|\cdot\|_{k,D}$, $|\cdot|_{k,D}$, and $\|\cdot\|_{0,D}$. We will drop D when $D = (0,1)$. Throughout this paper C denotes a generic positive constant that is independent of the mesh size.

2. Two Variants of Local Projection Stabilization

We consider the two-point boundary value problem

$$-\varepsilon u'' + bu' + cu = f \quad \text{in } (0,1), \qquad u(0) = u(1) = 0, \tag{1}$$

under the assumption

$$c - \frac{1}{2}b' \geq \gamma > 0, \tag{2}$$

which guarantees a unique weak solution $u \in H_0^1(0,1)$. Note that in the interesting case $0 < \varepsilon \ll 1$, the solution exhibits boundary and interior layers whose positions depend on the convection field b.