INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 3, Pages 535–548

A PARAMETER–UNIFORM FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED LINEAR DYNAMICAL SYSTEMS

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Dedicated to G. I. Shishkin on his 70th birthday

Abstract. A system of singularly perturbed ordinary differential equations of first order with given initial conditions is considered. The leading term of each equation is multiplied by a small positive parameter. These parameters are assumed to be distinct and they determine the different scales in the solution to this problem. A Shishkin piecewise–uniform mesh is constructed, which is used, in conjunction with a classical finite difference discretization, to form a new numerical method for solving this problem. It is proved that the numerical approximations obtained from this method are essentially first order convergent uniformly in all of the parameters. Numerical results are presented in support of the theory.

Key Words. linear dynamical system, multiscale, initial value problem, singularly perturbed, finite difference method, parameter–uniform convergence.

1. Introduction

We consider the initial value problem for the singularly perturbed system of linear first order differential equations

$$E\vec{u}'(t) + A(t)\vec{u}(t) = \vec{f}(t), \quad t \in (0, T], \quad \vec{u}(0) \text{ given.} \tag{1}$$

Here $\vec{u}$ is a column n-vector, E and $A(t)$ are $n \times n$ matrices, $E = diag(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1 \ldots n$. For convenience we assume the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

These n distinct parameters determine the n distinct scales in this multiscale problem. Cases with some of the parameters coincident are not considered here. We write the problem in the operator form

$$\vec{L}\vec{u} = \vec{f}, \quad \vec{u}(0) \text{ given,}$$

where the operator $\vec{L}$ is defined by

$$\vec{L} = ED + A(t) \text{ and } \mathrm{D} = \frac{\mathrm{d}}{\mathrm{dt}}.$$

We assume that, for all $t \in [0, T]$, the components $a_{ij}(t)$ of $A(t)$ satisfy the inequalities

Received by the editors August 24, 2009 and, in revised form, September 29, 2009.
1991 *Mathematics Subject Classification.* 65L05, 65L12, 65L20, 65L70.

$$a_{ii}(t) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(t)| \ \text{ for } \ 1 \le \mathrm{i} \le \mathrm{n}, \ \text{ and } \ \mathrm{a_{ij}(t)} \le 0 \ \text{ for } \ \mathrm{i} \neq \mathrm{j}. \tag{2}$$

We take α to be any number such that

$$0 < \alpha < \min_{\substack{t\in(0,1]\\ 1\le i\le n}} (\sum_{j=1}^{n} a_{ij}(t)). \tag{3}$$

We also assume that $T \ge 2\max_i(\varepsilon_i)/\alpha$, which ensures that the solution domain contains all of the layers. This condition is fulfilled if, for example, $T \ge 2/\alpha$. We introduce the norms $\| \vec{V} \| = \max_{1\le k\le n} |V_k|$ for any n-vector $\vec{V}$, $\| y \| = \sup_{0\le t\le T} |y(t)|$ for any scalar-valued function y and $\| \vec{y} \| = \max_{1\le k\le n} \| y_k \|$ for any vector-valued function $\vec{y}$. Throughout the paper C denotes a generic positive constant, which is independent of t and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

The plan of the paper is as follows. In the next section both standard and novel bounds on the smooth and singular components of the exact solution are obtained. The sharp estimates in Lemma 2.4 are proved by mathematical induction, while an interesting ordering of the points $t_{i,j}$ is established in Lemma 2.6. In Section 3 the appropriate piecewise-uniform Shishkin meshes are introduced, the discrete problem is defined and the discrete maximum principle and discrete stability properties are established. In Section 4 an expression for the local truncation error is found and two distinct standard estimates are stated. In Section 5 parameter-uniform estimates for the local truncation error of the smooth and singular components are obtained in a sequence of lemmas. The section culminates with the statement and proof of the parameter-uniform error estimate, which is the main theoretical result of the paper. In the final section numerical results are presented in support of the theory.

The initial value problems considered here arise in many areas of applied mathematics; see for example [1]. Parameter uniform numerical methods for simpler problems of this kind, when all the singular perturbation parameters are equal, were considered in [4]. A special case of the present problem, with $n = 3$, was considered in [3]. However, the proof of the parameter uniform error estimate for general n, which is the main goal of the present paper, is significantly more difficult. A general introduction to parameter uniform numerical methods is given in [2] and [7].

2. Analytical results

The operator $\vec{L}$ satisfies the following maximum principle

Lemma 2.1. *Let $A(t)$ satisfy (2) and (3). Let $\vec{\psi}(t)$ be any function in the domain of $\vec{L}$ such that $\vec{\psi}(0) \ge 0$. Then $\vec{L}\vec{\psi}(t) \ge 0$ for all $t \in (0, T]$ implies that $\vec{\psi}(t) \ge 0$ for all $t \in [0, T]$.*

Proof. Let i^*, t^* be such that $\psi_{i^*}(t^*) = \min_{i,t} \psi_i(t)$ and assume that the lemma is false. Then $\psi_{i^*}(t^*) < 0$. From the hypotheses we have $t^* \neq 0$ and $\psi'_{i^*}(t^*) \le 0$.