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A ROBUST FINITE DIFFERENCE METHOD FOR A SINGULARLY PERTURBED DEGENERATE PARABOLIC PROBLEM, PART I

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This paper is dedicated to Grisha Shishkin, on the occasion of his 70th birthday

Abstract. A singularly perturbed degenerate parabolic problem in one space dimension is considered. Bounds on derivatives of the solution are proved; these bounds depend on the two data parameters that determine how singularly perturbed and how degenerate the problem is. A tensor product mesh is constructed that is equidistant in time and of Shishkin type in space. A finite difference method on this mesh is proved to converge; the rate of convergence obtained depends on the degeneracy parameter but is independent of the singular perturbation parameter. Numerical results are presented.

Key Words. singularly perturbed, degenerate parabolic problem, Shishkin mesh

1. Introduction

Consider the singularly perturbed initial-boundary value problem

$$Lu(x,t) := \varepsilon u_{xx}(x,t) - x^\alpha u_t(x,t) = x^\alpha f(x,t) \quad \text{for } (x,t) \in \Omega, \tag{1a}$$

subject to the Dirichlet initial and boundary conditions

$$u(0,t) = \varphi_L(t) \quad \text{for } 0 < t \le T, \tag{1b}$$

$$u(x,0) = \varphi_0(x) \quad \text{for } 0 \le x \le 1, \tag{1c}$$

$$u(1,t) = \varphi_R(t) \quad \text{for } 0 < t \le T, \tag{1d}$$

where $\Omega := (0,1) \times (0,T]$ for some fixed $T > 0$, the small parameter $\varepsilon \in (0,1]$ and $\alpha > 0$ is a positive constant. The function f is smooth and the functions φ are continuous; further hypotheses will be placed on them later.

The differential operator L of (1) degenerates at the boundary $x = 0$ of $\bar{\Omega}$ and consequently its properties are not described by the standard theory of parabolic partial differential equations, even for fixed $\varepsilon > 0$. Thus (1) suffers from *two* distinct difficulties: its singularly perturbed nature (caused by the small parameter ε) and its degenerate nature (induced by the coefficient x^α of u_t).

At the boundary $x = 1$ the solution $u(x,t)$ displays a parabolic layer of width $O(\varepsilon^{1/2})$, as in the non-degenerate case, but a more complicated layer of width $O(\varepsilon^{1/(2+\alpha)})$ appears at the boundary $x = 0$. See Figure 1.

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Shishkin [6] studied the initial-boundary problem

$$Lu(x,t) = \varepsilon^{\alpha/(2+\alpha)} f_1(x,t) + x^\alpha f_2(x,t) \tag{2}$$

with the above Dirichlet data, where f_1 and f_2 are smooth. In a later paper [7] we shall consider this more general problem, which requires many changes in the analysis presented here. As mentioned in [6], problems like this arise when one models the transfer of heat over a rectangle in a medium moving with velocity x^α along the x-axis and conducting heat only across the flow; see also [5].

To solve (2) numerically, in [6] the author constructs a tensor product mesh with N_x points in the x direction and N_t points in the t direction. The x-mesh is a modified Shishkin-type mesh with three transition points while the t-mesh is equidistant. On this (x,t)-mesh a standard finite difference scheme is employed: central differencing in the x direction with backward differencing in the t direction. Writing u^N for the numerical solution, it is shown in [6] that the maximum nodal error in $u - u^N$, measured uniformly in ε, are

$$\mathcal{O}(N_x^{-1} \ln N_x + N_t^{-1}) \quad \text{for } 1 \le \alpha \le 2,$$
$$\mathcal{O}(N_x^{-1} \ln N_x + N_x^{-4/(2+\alpha)} + N_t^{-1}) \quad \text{for } \alpha > 2,$$

but the presentation is very concise and consequently some arguments are unclear.

When (2) is replaced by the simpler problem (1), an inspection of [6] shows that one of the mesh transition points can be omitted and the maximum nodal error in $u - u^N$, measured uniformly in ε, is now $\mathcal{O}(N_x^{-1} \ln N_x + N_t^{-1})$ for all $\alpha \ge 1$.

In the present paper we sharpen this result of [6] by showing that in fact for (1) the maximum nodal error, measured uniformly in ε, is $\mathcal{O}(N_x^{-2}(\ln N_x)^2 + N_t^{-1})$ if $\alpha = 1$ or $\alpha \ge 2$. For completeness we also prove the bound $\mathcal{O}(N_x^{-1} \ln N_x + N_t^{-1})$ for $1 < \alpha < 2$. All our arguments are given in detail. Numerical results will be presented to illustrate the accuracy of the numerical method.

Notation. We use C to denote a generic constant that is independent of ε and of any mesh used. Thus C can take different values in different places, even in the same calculation. Set $S(\Omega) = \bar{\Omega} \setminus \Omega$; this is the set of points where the initial and boundary conditions are prescribed. The space of continuous functions defined on any measurable subset ω of Ω is $C(\omega)$ and the $L_\infty(\omega)$ norm on $C(\omega)$ is denoted by $\|\cdot\|_\omega$, except that when $\omega = \Omega$ we simply write $\|\cdot\|$. For non-negative integers m, k and measurable $\omega \subset \Omega$, a function g is said to lie in $C^{m,k}(\omega)$ if $\partial^{i+j} g/\partial x^i \partial t^j \in C(\omega)$ for $0 \le i \le m$ and $0 \le j \le k$.

Finally, set

$$\nu = \frac{1}{2+\alpha} \quad \text{and} \quad \gamma = \frac{\alpha}{2+\alpha}\,.$$

Note that $\gamma = \alpha\nu$ and $2\nu + \gamma = 1$.

2. Properties of the solution u of (1)

By a standard argument [3, Section 2.1] one sees that the differential operator L satisfies the usual maximum principle:

Lemma 1. *Let $\Psi \in C(\bar{\Omega}) \cap C^{2,1}(\Omega)$ with $\Psi \ge 0$ on $S(\Omega)$. If $L\Psi \le 0$ on Ω then $\Psi \ge 0$ on $\bar{\Omega}$.*

This lemma can be used to bound u via a barrier function Φ:

Lemma 2. *Assume that u and Φ lie in $C(\bar{\Omega}) \cap C^{2,1}(\Omega)$ with $\Phi \ge |u|$ on $S(\Omega)$ and $L\Phi \le -Lu$ on Ω. Then $|u| \le \Phi$ on $\bar{\Omega}$.*

Proof. Apply Lemma 1 to the functions $\Phi \pm u$ to get $\Phi \pm u \ge 0$ on $\bar{\Omega}$. □