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A UNIFORM NUMERICAL METHOD FOR A BOUNDARY-SHOCK PROBLEM

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This paper is dedicated to G. I. Shishkin.

Abstract. A singularly perturbed quasilinear boundary-value problem is considered in the case when its solution has a boundary shock. The problem is discretized by an upwind finite-difference scheme on a mesh of Shishkin type. It is proved that this numerical method has pointwise accuracy of almost first order, which is uniform in the perturbation parameter.

Key Words. Boundary-value problem, singular perturbation, boundary shock, finite differences, Shishkin mesh, uniform convergence.

1. Introduction

Consider the problem of finding a $C^2(0,1)$-function $u = u(x)$ which solves the following singularly perturbed boundary-value problem:

$$-\varepsilon u'' - ub(u)u' + uc(x,u) = 0, \quad x \in (0,1), \quad u(0) = 0, \quad u(1) = B, \tag{1}$$

where $' = \mathrm{d}/\mathrm{d}x$, ε is a small positive perturbation parameter, and B is a positive constant. It is assumed that the functions b and c are sufficiently smooth and satisfy certain conditions, the main ones being $b > 0$ and $c_u \geq 0$. All the assumptions are specified in section 2. They are exactly the same as in [13] and they guarantee that there exists a unique solution u of problem (1) and that u has an exponential boundary layer at $x = 0$.

In [13], (1) is solved numerically by applying a layer-resolving transformation which renders the derivatives of the transformed solution bounded uniformly in ε. The transformed problem is then solved using finite-difference schemes on an equidistant discretization mesh. The layer-resolving transformation corresponds to mesh-generating functions used to create special meshes, dense in the boundary layer, for discretizing the problem (1) directly, cf. [16]. Numerical results obtained by this method show pointwise ε-uniform convergence. However, only L^1 ε-uniform convergence is proved in [13]. The same result is obtained in [18], but for an exponentially-fitted equidistant finite-difference scheme and for a special case ($b \equiv 1$) of problem (1). This special case has been recently considered in [17], where a robust error estimate in the maximum norm is derived. This is achieved by applying the approach in which the differential equation

$$-\varepsilon u'' - \frac{1}{2}(u^2)' + uc(x,u) = 0 \tag{2}$$

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is integrated from x to 1 and then the integral $\int_x^1 u(t)c(t,u(t))\mathrm{d}t$ is approximated using the solution of the corresponding reduced problem, cf. [6, 5]. After the described transformation, equation (2) becomes a Riccati equation, which is solved by the method from [11]. This method uses the simple backward scheme on a Shishkin-type mesh. The error of the approximate solution obtained in this way can be estimated at each mesh point by

$$M[\varepsilon + N^{-1}(\ln N)^2], \tag{3}$$

where N is the number of mesh steps and M is a positive constant independent of both ε and N. Since it often holds in practice that $\varepsilon \ll 1/N$, this error estimate gives accuracy of almost first order (*almost* means here that the accuracy is diminished by $\ln N$ factors). Nevertheless, strictly speaking, (3) does not mean convergence uniform in ε. This result can still be used to achieve ε-uniform convergence, but the method has to be combined with some classical method for solving differential equations, see the discussion in [6, 5]. However, the order of ε-uniform convergence resulting from the combination is lowered since the error can be estimated by $MN^{-\omega}$ with $0 < \omega < 1$. The goal of the present paper is to prove that ε-uniform convergence of order almost 1 can be achieved.

In the numerical method considered here, contrary to [13, 17], the only transformation of the problem is to its conservation form which is then discretized by an upwind finite-difference scheme on a Shishkin piecewise equidistant mesh. There is nothing new about this numerical method, but its analysis is new. Crucial in this is the technique from [9] used to discuss the stability of the discretization scheme. It is originally applied in [9] to a semilinear singular perturbation problem with a boundary turning point. The technique is here adjusted to the quasilinear problem and relies heavily on the Shishkin mesh used. The result is the pointwise error-estimate of the form $MN^{-1}(\ln N)^3$.

The problem (1) can be referred to as a *boundary-shock problem* in contrast to interior-shock problems for which the boundary condition at $x = 0$ is $u(0) = A < 0$, see [4] and [10] for instance. The difficulty in trying to obtain ε-uniform pointwise accuracy for interior-shock problems lies in the fact that the interior shock of the numerical solution is shifted from the original location. The method of the present paper can be applied to interior-shock problems only if the position of the shock is known; then the interior-shock problem can be broken down to two problems of type (1).

The rest of the paper is organized as follows. Properties of the continuous solution are given in section 2, which is based on [13]. The numerical method is described in section 3 and the main result is also proved there. Finally, section 4 contains some numerical results which illustrate the previously presented theory.

2. The continuous problem

The problem (1) is discretized in its conservation form,

$$Tu := -\varepsilon u'' - f(u)' + g(x,u) = 0, \quad x \in (0,1), \quad Ru := (u(0), u(1)) = (0, B), \tag{4}$$

where $B > 0$,

$$f(u) = \int_0^u tb(t)\,\mathrm{d}t, \quad \text{and} \quad g(x,u) = uc(x,u).$$

Although usually ε is small, a wider range of ε values is considered, $\varepsilon \in (0,1]$.

Detailed conditions on b and c follow, cf. [13]. Let $X = [0,1]$ and $U = [0,B]$. It is assumed that $b \in C^2(U)$ and $c \in C^2(X \times U)$ since this is needed for the proof of