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TWO-GRID ALGORITHMS FOR AN ORDINARY SECOND ORDER EQUATION WITH AN EXPONENTIAL BOUNDARY LAYER IN THE SOLUTION

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Dedicated to G.I.Shishkin on the occasion of his 70th birthday

Abstract. This paper is concerned with the solution of the nonlinear system of equations arising from the A.M. Il'in's scheme approximation of a model semilinear singularly perturbed boundary value problem. We employ Newton and Picard methods and propose a new version of the two-grid method originated by O. Axelsson [2] and J. Xu [19]. In the first step, the nonlinear differential equation is solved on a "coarse" grid of size H. In the second step, the problem is linearized around an appropriate interpolation of the solution computed in the first step and the linear problem is then solved on a fine grid of size $h << H$. It is shown that the algorithms achieve optimal accuracy as long as the mesh sizes satisfy $h = O(H^{2^m}),\ m = 1, 2, \dots,$ where m is the number of the Newton (Picard) iterations for the difference problem. We count the number of the arithmetical operations to illustrate the *computational cost* of the algorithms. Numerical experiments are discussed.

Key Words. nonlinear boundary value problem, boundary layer, Il'in scheme, nonlinear system, Newton method, Picard method, two-grid method.

1. Introduction

It is shown theoretically and experimentally that classical finite difference schemes on non-adaptive meshes have a cell Reynolds number limitation when applied to convection-dominated equations [3,6,7, 8,10,11]. For small values of the perturbation parameter, these techniques lead to spurious solutions when central differences for the advection terms are employed; on the other hand, first-order upwind methods introduce artificial diffusion that thickens the boundary layers. In order to avoid these difficulties, exponentially-fitting techniques are frequently used [1,6,7,8]. Another approach is based on the generation of layer-adapted meshes that allow resolution of the structure of the layer [3,10,11,12].

The defect correction and the Richardson extrapolation methods are used to increase the accuracy of grid solutions for singularly perturbed boundary value problems. Note that the nonlinear case has been considered in [12]. However, the Richardson procedure requires the solution of discrete nonlinear systems on each of the nested meshes.

Two-level discretizations can be dated back to Allen-Southwell [1], see also [5]. In the present paper we shall develop a new version based on the quasilinearization

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method of Belman and Kalaba [4], see also [9]. Two-grid finite element methods were proposed by O. Axelsson [2] and J. Xu [19], independently of each other. Note that the error estimates in these papers are in *weak (Sobolev-type)* discrete norms. Conversely, our errors below are measured in the *maximum* norm, which is sufficiently strong to capture layers and hence seems most appropriate for singularly perturbed problems.

We illustrate some of these concepts on the model boundary value problem

$$-\varepsilon u'' - a(x)u' + f(x,u) = 0,\ x \in \Omega \equiv (0,1);\ u(0) = A,\ u(1) = B, \tag{1}$$

where A, B are given constants, ε is a parameter in $(0,1]$, and $a(x)$ satisfies

$$|a(x)| \geq \alpha > 0,\ a \in C^2(\overline{\Omega}). \tag{2}$$

For the function $f(x,u)$ we will assume that it is twice continuously differentiable with respect to x , three times continuously differentiable with respect to u and

$$f'_u(x,u) \geq 0 \text{ on } \Omega \times R. \tag{3}$$

By these assumptions the problem (1) has the unique solution $u = u(x,\varepsilon)$ and has a boundary layer of order $O(\varepsilon)$ near $x = 0$ or $x = 1$, see for example [8,11,16].

The goal of the present paper is to construct and study theoretically and numerically two-grid interpolation algorithms for implementation of the classical Il'in's difference scheme [6] for problem (1)-(3). We begin by recalling in the next section basic properties of problem (1)-(3) and an already classical uniformly convergent result for the corresponding linear problem, Theorem 1. Then, in Section 3, we describe a Newton linearization process for the differential problem (1)-(3) in order not only to prove uniform convergence of Il'in's scheme but first of all to obtain the estimate (18) which is the key for the two-grid algorithms in the next sections. In Sections 4, 5 we employ Newton and Picard methods in the solution of the arizing systems of algebraic equations. The two-grid algorithms are formulated and their rate of convergence is established in Section 6. This strategy is motivated by the fact (Theorem 3) that the global error of the two-grid interpolation algorithm is of the order h, the same as would have been obtained if the non-linear problem had been solved directly on the fine grid. The coarse mesh can be quite coarse, (see the experiments in Section 7) and still maintain an optimal approximation.

Part of the present results was published in the conference paper [14].

Notation. Define the norm of a continuous function $f(x)$ as $\|f\| = \max_{x\in\overline{\Omega}} |f(x)|$. Throughout this paper C and $C_i,\ i \geq 0$, denote positive constants independent of H, h and ε . If $z = (z_0, \ldots, z_N) \in R^{N+1}$ is a mesh function, define its discrete norm as $\|z\|_h = \max_{0\leq i\leq N} |z_i|$. For a continuous function f defined on Ω by $[f]_{\overline{w}_h}$ we will denote it's projection on a mesh $\overline{w}_h \subset \overline{\Omega}$. In the text $u, u^{(m)}$ and $y, y^{(m)}$ denote continuous and discrete functions, respectively.

2. Preliminary analysis

In the following we will consider the problem (1)-(3) in the case $a(x) \geq \alpha > 0$. The other case $a(x) \leq \alpha < 0$ can be put into the form of the first case by the change of the independent variable from x to $1 - x$.

At first, we get the estimate for the solution of the problem (1)-(3):

$$\|u\| \leq l = \alpha^{-1}\|f(x,0)\| + \max\{|A|,|B|\}. \tag{4}$$

Let us introduce the linear operator:

$$Gz(x) = -\varepsilon z''(x) - a(x)z'(x) + b(x)z(x),$$