INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 4, Pages 619–638

WEAKLY NONLINEAR ANALYSIS OF THE HAMILTON–JACOBI–BELLMAN EQUATION ARISING FROM PENSION SAVINGS MANAGEMENT

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(Communicated by Lubin G. Vulkov)

Abstract. The main purpose of this paper is to analyze solutions to a fully nonlinear parabolic equation arising from the problem of optimal portfolio construction. We show how the problem of optimal stock to bond proportion in the management of pension fund portfolio can be formulated in terms of the solution to the Hamilton–Jacobi–Bellman equation. We analyze the solution from qualitative as well as quantitative point of view. We construct useful bounds of solution yielding estimates for the optimal value of the stock to bond proportion in the portfolio. Furthermore we construct asymptotic expansions of a solution in terms of a small model parameter. Finally, we perform sensitivity analysis of the optimal solution with respect to various model parameters and compare analytical results of this paper with the corresponding known results arising from time-discrete dynamic stochastic optimization model.

Key Words. Hamilton–Jacobi–Bellman equation, weakly nonlinear analysis, asymptotic expansion, fully nonlinear parabolic equation, stochastic dynamic programming, pension savings accumulation model.

1. Introduction and problem formulation

In this paper we are analyzing solutions to the Hamilton–Jacobi–Bellman equation arising from stochastic dynamic programming for optimal decision between stock and bond investments during accumulation of pension savings. Such an optimization problem often arises in optimal dynamic portfolio selection and asset allocation policy for an investor who is concerned about the performance of a portfolio relative to the performance of a given benchmark (see e.g. [18, 19, 20, 21, 5, 3, 4, 8, 10]).

Consider the function $V(t,y), (t,y) \in \mathcal{D}$, defined on a domain $\mathcal{D} = [0,T) \times (0,\infty)$ and satisfying the following fully nonlinear Hamilton–Jacobi–Bellman parabolic partial differential equation:

$$\frac{\partial V}{\partial t} + \max_{\theta \in \Delta_t} \left(A_\varepsilon(\theta,t,y)\frac{\partial V}{\partial y} + \frac{1}{2}B^2(\theta,t,y)\frac{\partial^2 V}{\partial y^2}\right) = 0, \qquad (t,y) \in \mathcal{D}, \tag{1a}$$

and the terminal condition at $t = T$,

$$V(T,y) = U(y), \quad y \in (0,\infty), \tag{1b}$$

Received by the editors April 30, 2009, and in revised form, November 5, 2009 .

2000 *Mathematics Subject Classification.* 35K55 34E05 70H20 91B70 90C15 91B16.

This research was supported by VEGA 1/0381/09 (CESIUK project) and bilateral Slovak–Bulgarian project APVV SK-BG-0034-08.

where $U = U(d)$ is a smooth strictly increasing concave bounded function and ε is a small parameter, $0 < \varepsilon \ll 1$. Moreover, we suppose that the following additional requirements are met:

(1) the admissible set $\Delta_t = [l_t, u_t] \subset \mathbb{R}$ for all $0 \le t \le T$;
(2) the function $\Delta_t \ni \theta \mapsto A_\varepsilon(\theta, t, y) \in \mathbb{R}$ is (not necessarily strictly) concave in the θ variable and it is function increasing at $\theta = l_t$;
(3) the function $\Delta_t \ni \theta \mapsto B^2(\theta, t, y)$ is strictly convex in the θ variable and it is decreasing at $\theta = l_t$.

Let us suppose for a moment that the function $y \mapsto V(t, y)$ is an increasing and strictly concave function in the y variable. Then applying the first order necessary condition on the maximum of the function

$$\theta \mapsto A_\varepsilon(\theta, t, y)\frac{\partial V}{\partial y} + \frac{1}{2}B^2(\theta, t, y)\frac{\partial^2 V}{\partial y^2}$$

we obtain the following implicit equation for $\hat{\theta}$, the maximizer of the above function:

$$G(\hat{\theta}, t, y) = -\frac{\frac{\partial V}{\partial y}(t, y)}{y\frac{\partial^2 V}{\partial y^2}(t, y)} \quad \text{where} \quad G(\theta, t, y) = \frac{1}{2}\frac{\frac{\partial (B^2)}{\partial \theta}}{y\frac{\partial A_\varepsilon}{\partial \theta}}. \tag{2}$$

Since the requirements (2)–(3) guarantee the increase of the function $G(\theta, t, y)$ in the θ variable, there exists the inverse of G and thus the unique $\hat{\theta} = \tilde{\theta}(t, y)$ such that

$$\hat{\theta}(t, y) = G^{-1}\left(-\left(\frac{\partial V}{\partial y}(t, y)\right) / \left(y\frac{\partial^2 V}{\partial y^2}(t, y)\right)\right).$$

Then the optimal value of θ solving (1a) with the terminal condition (1b) is given by

$$\theta^*(t, y) = \min\{u_t, \hat{\theta}(t, y)\}. \tag{3}$$

The problem (1a) can be now treated as a fully nonlinear parabolic partial differential equation of the form:

$$\frac{\partial V}{\partial t} + \mathcal{F}(t, y, V, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial y^2}) = 0 \tag{4a}$$

where

$$\mathcal{F}(t, y, V, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial y^2}) = A_\varepsilon(\theta^*(t, y), t, y)\frac{\partial V}{\partial y} + \frac{1}{2}B^2(\theta^*(t, y), t, y)\frac{\partial^2 V}{\partial y^2} \tag{4b}$$

where θ^* is given by (3) and $\hat{\theta}(t, y) = G^{-1}\left(-\left(\frac{\partial V}{\partial y}(t, y)\right) / \left(y\frac{\partial^2 V}{\partial y^2}(t, y)\right)\right)$ depends itself on the solution V and its derivatives. The solution is subject to the terminal condition $V(T, y) = U(y)$ where $V = V(t, y)$ for $y > 0$ and $0 \le t \le T$. Furthermore, $\frac{\partial \mathcal{F}}{\partial q} > 0$.

The application of this study to financial markets, particularly to the theory of the optimal portfolio construction, has a strong impact on the special choice of the functions A_ε and B used in the original formulation of the studied problem (1a). Hence let us consider

$$A_\varepsilon(\theta, t, y) = \varepsilon + [\mu_t(\theta) - \beta_t]y, \qquad B(\theta, t, y) = \sigma_t(\theta)y, \tag{5}$$

where $\{\beta_t; 0 \le t \le T\}$, $\{\mu_t(\theta); 0 \le t \le T\}$ and $\{\sigma_t(\theta); 0 \le t \le T\}$ are assumed to be given deterministic processes for any choice of the control parameter $\theta \in \Delta_t$.

Moreover, if $V(t, y)$ is strictly convex and increasing in the y variable, then with regard to assumptions (2)–(3) the monotonicity of the function $\theta \mapsto G(\theta, t, y)$ is