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SUBGRID MODEL FOR THE STATIONARY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS BASED ON THE HIGH ORDER POLYNOMIAL INTERPOLATION

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Abstract. In this paper, we propose a subgrid finite element method for the two-dimensional (2D) stationary incompressible Naver-Stokes equation (NSE) based on high order finite element polynomial interpolations. This method yields a subgrid eddy viscosity which does not act on the large scale flow structures. The proposed eddy viscous term consists of the fluid flow fluctuation stress. The fluctuation stress can be calculated by means of simple reduced-order polynomial projections. Assuming some regular results of NSE, we give a complete error analysis. Finally, in the part of numerical tests, the numerical computations show that the numerical results agree with some benchmark solutions and theoretical analysis very well.

Key Words. Navier-Stokes equation, subgrid method, eddy viscosity, error analysis and numerical tests.

1. Introduction

In this paper, we focus on formulating a subgrid eddy viscosity method for the stationary incompressible Navier-Stokes equation. For the subgrid method, we must admit that there exists a scale separation between large and small scales. This model can be viewed as a viscous correction for large scale fluid flows. For the laminar fluid flows, the added subgrid viscosity term should not affect the large scale structures of fluid flow fields and should tend to vanish. These kinds of subgrid methods are flexible and effective for high Reynolds number fluid flows.

It is well-known that for most problems of fluid flows, the numerical algorithms capturing all scales of fluid flows are impossible. In complex fluid flows, there often exist several scales that span from the large scales to the small Kolmogorov scales which hardly be resolved by state-of-the-art computers for most engineering problems. Especially, for the convection-dominated fluid flows, we often need to consider the dispersive effects of unresolved scales on resolved scales. The eddy viscosity models are often utilized to model and solve this kind of problems by engineers, which have been achieved many successes in engineering practice [1]. These kinds of models are firstly proposed by Boussinesq [2], developed by Taylor and Prandlt [3], and introduced a dissipative mechanism by Smagorinsky [4]. At present, these models have been further improved by various numerical methods [5, 6, 7]. In existing mathematical models, these eddy viscosity models are established by introducing the scale separation based on L^2 and elliptic projection. Recently,

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Hughes *et al* has proposed a variational multi-scale method (VMM) in which the diffusion acts only on the finest resolved scales. This VMM is very effective to model this complex multi-scale phenomena. The key problem focuses on introducing a reasonable scale separation (coarse and fine scales). Generally, there exist many different ways to define coarse and fine scales according to the VMM framework [8]. According to the idea of VMM, the subgrid methods in this paper are variational multiscale methods.

In this paper, we will implement a subgrid method to remove the dispersive effects from small scales by virtue of low-order polynomial projections. The added subgrid term does not need special treatments for implementing calculations. The added subgrid term is calculated by simple treatments of basis functions, which will be given in the section of numerical tests. And you can find an analogous treatment in [9]. But, the method in [9] is based on a projection from a fine finite element space to a coarse finite element space.

The adopted finite element pair is the P_2/P_1 pair to approach velocity-pressure fields. For low Reynolds number fluid flows, the results indicate that this method has a convergence rate of the same order as the standard Garlerkin method. By the numerical tests, it is shown that the proposed subgrid correction model can simulate the fluid flows correctly and does not act on the large scale flow structures.

The outline of the paper is organized as follows. In the next section we introduce the Navier-Stokes equations (NSE) and the corresponding function settings. In section 2, we give the NSE and the corresponding functional settings. In section 3, the subgrid viscous term is introduced into the NSE and the standard Galerkin discretization of the Navier-Stokes equations is given. In section 4, we show the results of the error estimates. Some numerical results are presented in section 5, which show the correctness and efficiency of the methods. Finally, we give some conclusions.

2. Navier-Stokes equations and functional settings

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. We consider the stationary Navier-Stokes equation

$$\left\{ \begin{aligned} -\nu\Delta u + \nabla p + (u\cdot\nabla)u &= f, \quad \text{in } \Omega \\ \text{div } u &= 0, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \Gamma \end{aligned} \right. \tag{1}$$

where $u = (u_1, u_2)$ represents the velocity vector, p denotes the pressure, f is the body force and $\nu > 0$ is the viscosity.

We introduce the following functional settings

$$X := H_0^1(\Omega)^2, V := \{v \in X, \text{div} v = 0\}, Y := (L^2(\Omega))^2$$
$$Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \textstyle\int_\Omega q dx = 0\}.$$

We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ are the inner product and norm in $L^2(\Omega)$ or $L^2(\Omega)^2$. The space $H^k(\Omega)$ or $H^k(\Omega)^2$ denotes the standard Sobolev spaces with norm $\|\cdot\|_k$ and semi-norm$|\cdot|_k$. The space $H_0^1(\Omega)$ or $H_0^1(\Omega)^2$ is equipped with the following scalar product and norm

$$((u, v)) = (\nabla u, \nabla v), |u|_1 = ((u, u))^{1/2}.$$

The space $H^{-1}(\Omega)^2$ is the dual space of $H_0^1(\Omega)^2$ equipped with the norm

$$\|z\|_{-1} = \sup_{v\in H_0^1(\Omega)^2} \frac{|(z, v)|}{|v|_1}$$