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ANALYSIS OF AN INTERACTION PROBLEM BETWEEN AN ELECTROMAGNETIC FIELD AND AN ELASTIC BODY

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Abstract. This paper deals with an interaction problem between a solid and an electromagnetic field in the frequency domain. More precisely, we aim to compute both the magnetic component of the scattered wave and the elastic vibrations that take place in the solid elastic body. To this end, we solve a transmission problem holding between the bounded domain $\Omega_s \subset \mathbb{R}^3$ representing the obstacle and a sufficiently large annular region surrounding it. We point out here that (following Voigt's model, cf. [12]) we only allow the electromagnetic field to interact with the elastic body through the boundary of Ω_s. We apply the abstract framework developed in the work [3] by A. Buffa to prove that our coupled variational formulation is well posed. We define the corresponding discrete scheme by using the edge element in the electromagnetic domain and standard Lagrange finite elements in the solid domain. Then we show that the resulting Galerkin scheme is uniquely solvable, convergent and we derive optimal error estimates. Finally, we illustrate our analysis with some results from computational experiments.

Key Words. edge finite elements, Maxwell equations and elastodynamics equations.

1. Introduction

In this paper we develop a finite element method for a time-harmonic problem that models the interaction between an elastic body and an electromagnetic field. We consider a solid occupying a bounded region $\Omega_s \subset \mathbb{R}^3$ and assume that it is subject to a given incident electromagnetic wave. Actually, we suppose here that the electromagnetic field occupies an annular region Ω_m whose exterior boundary Γ is located far from the obstacle (the solid body) and impose on this artificial closed surface a boundary condition compatible with the behavior of the scattered field at infinity. Moreover, we assume that the penetration of the electromagnetic field inside the body is not large enough to consider. The interaction between the electromagnetic field and the elastic body is only governed by the equilibrium of tangential forces on the interface $\Sigma := \partial\Omega_s$. This model problem is a simplification of the one presented by Cakoni and Hsiao in [7]. To the best of our knowledge, the numerical study of this interaction problem has not been treated in the literature. Our aim is to provide a finite element Galerkin scheme that permits one to compute both the scattered electromagnetic wave and the elastic vibrations of the solid.

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Once the variational formulation is derived, it becomes clear that the term coupling the elastodynamics equation in Ω_s and Maxwell equations in Ω_m is a compact perturbation (see (29) below). This means that we are almost left with an separate study of both equations in each domain. The primal formulation of the elasticity problem in Ω_s is standard. The operator arising from the corresponding primal formulation of the elastodynamics equation fails to be elliptic due to the "wrong" sign of the zero order term. Nevertheless, the compactness of the embedding $H^1(\Omega_s) \hookrightarrow L^2(\Omega_s)$ allows one to use successfully a Fredholm alternative to analyze its solvability.

The Maxwell problem is more intricate since it does not fit in any classical theory for proving well-posedness. Actually, since the canonical embedding $\mathbf{H}(\mathbf{curl}\,,\Omega_m) \hookrightarrow [L^2(\Omega_m)]^3$ is not compact, it is not possible to employ a Fredholm alternative, at least for the original form of the resulting variational formulation. The difficulty is then related to the noncoerciveness of the sesquilinear form arising in the study of Maxwell equations (written here in terms of the magnetic field). A Helmholtz-type decomposition of the magnetic field is usually proposed in order to reveal hidden compactness properties that permits to deal with the study of this problem through a classical analysis, see [3, 13] and the references cited therein. Actually, Buffa [3] succeeded in setting up this technique in a general abstract framework. We follow here this technique, our analysis is based on a suitable decomposition of $\mathbf{H}_\Gamma(\mathbf{curl}\,,\Omega_m)$ (see (18) below) that renders possible the application of a Fredholm alternative to the whole coupled problem. The corresponding discrete scheme is defined with the first order Nédélec element (also known as the edge element) in the electromagnetic domain and traditional first order Lagrange finite elements in the solid. The stability and convergence of this Galerkin method also relies on a stable decomposition of the finite element space used to approximate the magnetic field.

The remainder of the paper is organized as follows. In the next section we recall some essential tools related with tangential trace operators in the space $\mathbf{H}(\mathbf{curl}\,,\Omega)$. In sections 3 and 4 we give a brief description of the model problem and derive its coupled variational formulation. In section 5, we use a Fredholm alternative to show that, under some regularity conditions on the coefficients, the problem is well-posed. The corresponding Galerkin scheme is analyzed in section 6. Finally, in section 7 we provide results from numerical experiments that confirm our theoretical assertions.

We end this section with some notations to be used below. Since in the sequel we deal with complex valued functions, we let $\mathbb{C}$ be the set of complex numbers, use the symbol $\imath$ for $\sqrt{-1}$, and denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$. In addition, given any Hilbert space U, $[U]^3$ denotes the space of vectors of order 3 with entries in U. Given $\boldsymbol{\sigma} := (\sigma_{ij})$, $\boldsymbol{\tau} := (\tau_{ij}) \in \mathbb{C}^{3\times 3}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^3 \tau_{ii}$ and the tensor product $\boldsymbol{\sigma} : \boldsymbol{\tau} := \sum_{i,j=1}^3 \sigma_{ij}\,\tau_{ij}$. Finally, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector, and use C , with or without subscripts, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. Preliminaries

We denote by $\Omega \subset \mathbb{R}^3$ a generic bounded polyhedral domain and let $\boldsymbol{n}$ be the outward normal vector on its boundary Σ. We recall that

$$\mathbf{H}(\mathbf{curl}\,,\Omega) := \left\{ \boldsymbol{w} \in [L^2(\Omega)]^3; \quad \mathbf{curl}\, \boldsymbol{w} \in [L^2(\Omega)]^3 \right\}$$