INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 4, Pages 785–805

OPERATOR SPLITTING METHODS FOR THE NAVIER-STOKES EQUATIONS WITH NONLINEAR SLIP BOUNDARY CONDITIONS

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(Communicated by Zhangxin Chen)

Abstract. In this paper, the θ scheme of operator splitting methods is applied to the Navier-Stokes equations with nonlinear slip boundary conditions whose variational formulation is the variational inequality of the second kind with the Navier-Stokes operator. Firstly, we introduce the multiplier such that the variational inequality is equivalent to the variational identity. Subsequently, we give the θ scheme to compute the variational identity and consider the finite element approximation of the θ scheme. The stability and convergence of the θ scheme are showed. Finally, we give the numerical results.

Key Words. Navier-Stokes Equations, Nonlinear Slip Boundary Conditions, Operator Splitting Method, $\theta-$Scheme, Finite Element Approximation.

1. Introduction

Numerical simulation for the incompressible flow is the fundamental and significant problem in computational mathematics and computational fluid mechanics. It is well known that the mathematical model of viscous incompressible fluid with homogeneous boundary conditions is the Navier-Stokes equations

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ \text{div}u = 0. \end{cases}$$

It is obvious that (1) is a coupled system with a first-order nonlinear evolution equation and an imposed incompressible constrain so that the numerical simulation for the Navier-Stokes equations is very difficult. The popular technique to overcome this difficulty is to relax the solenoidal condition in an appropriate method and to result in a pesudo-compressible system, such as the penalty method and the artificial compressible method. The operator splitting method is also very useful to overcome this shortage. The main advantage is that it can decouple the difficulties associated to the nonlinear property with those associated to the incompressible condition. For more detail, see [1].

The operator splitting method has been a popular tool for the numerical simulation of the incompressible viscous flow. Based on the main idea of the operator splitting method, there have some different schemes, such as the Peaceman-Rachford

Received by the editors July 7, 2009 and, in revised form, April 19, 2010.
2000 *Mathematics Subject Classification.* 35Q30, 65N12.
This research was supported by the National Natural Science Foundation of China (No.10901122 and No.10971165) and by Zhejiang Provincial Natural Science Foundation of China (Y6090108).

scheme [2], the Douglas-Rachford scheme [3] and the θ scheme [4-5]. In this paper, we only apply the θ scheme to the Navier-Stokes equations with nonlinear slip boundary conditions. This class of boundary conditions are introduced by Fujita in [6-7], where he investigated some hydrodynamics problems under nonlinear boundary conditions, such as leak and slip boundary conditions involving a subdifferential property. These types of boundary conditions appear in the modeling of blood flow in a vein of an arterial sclerosis patient and in that of avalanche of water and rocks. Moreover, the variational formulation of the Navier-Stokes equations with these nonlinear boundary conditions is the variational inequality of the second kind.

The stability analysis of the θ scheme for the Navier-Stokes equations with the whole homogeneous Dirichlet boundary conditions has been investigated in [8]. The difficulty lies in the treatment of the trilinear term in the right-hand side. However, in this paper, besides the trilinear term, another difficulty is due to that the variational formulation is the variational inequality. To overcome this difficulty, we introduce the multiplier such that the variational inequality is equivalent to the variational identity.

2. The Navier-Stokes Equations

Consider the following Navier-Stokes equations:

$$
\left\{ \begin{array}{ll} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & \text{in } Q_T, \\ \mathrm{div} u = 0, & \text{in } Q_T, \end{array} \right. \tag{2}
$$

where $Q_T = \Omega\times[0,T]$ for some $T>0$, $u(t,x)$ denotes the velocity, $p(t,x)$ denotes the pressure, $f(t,x)$ denotes the external force and $\nu>0$ is the kinematic viscous coefficient. The domain $\Omega\subset\mathbb{R}^2$ is a bounded domain.

Given the initial value $u(0,x)=u_0(x)$ in Ω, we consider the following nonlinear slip boundary conditions:

$$
\left\{ \begin{array}{ll} u = 0 & \text{on } \Gamma\times(0,T], \\ u_n = 0, \quad -\sigma_\tau(u)\in g\partial|u_\tau| & \text{on } S\times(0,T], \end{array} \right. \tag{3}
$$

where $\Gamma\cap S=\emptyset, \overline{\Gamma\cup S}=\partial\Omega$ with $|\Gamma|\neq 0, |S|\neq 0$. g is a scalar functions; $u_n = u\cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, where n stands for the unit vector of the external normal to S; $\sigma_\tau(u)=\sigma-\sigma_n n$, independent of p, is the tangential component of the stress vector σ which is defined by $\sigma_i=\sigma_i(u,p)=(\mu e_{ij}(u)-p\delta_{ij})n_j$, where $e_{ij}(u)=\frac{\partial u_i}{\partial x^j}+\frac{\partial u_j}{\partial x^i}, i,j=1,2$. The set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a:

$$
\partial\psi(a)=\{b\in\mathbb{R}^2 : \psi(h)-\psi(a)\geq b\cdot(h-a), \quad \forall\ h\in\mathbb{R}^2\}.
$$

Denote

$$
V=\{u\in H^1(\Omega)^2;\ u|_\Gamma=0,\ u\cdot n|_S=0\}, \quad V_0=H_0^1(\Omega)^2; \quad V_\sigma=\{u\in V\mid \mathrm{div}u=0\};
$$

$$
H=\{u\in L^2(\Omega)^2;\ u\cdot n|_{\partial\Omega}=0\}, \quad M=L_0^2(\Omega)=\{q\in L^2(\Omega);\ \int_\Omega q\mathrm{d}x=0\}.
$$

Let $||\cdot||_k$ be the norm of the Hilbert space $H^k(\Omega)$ or $H^k(\Omega)^2$. Let $(\cdot,\cdot)$ and $||\cdot||$ be the inner product and the norm in $L^2(\Omega)^2$ or $L^2(\Omega)$. Then we can equip the inner product and the norm in V by $(\nabla\cdot,\nabla\cdot)$ and $||\cdot||_V=||\nabla\cdot||$, respectively, because $||\nabla\cdot||$ is equivalent to $||\cdot||_1$ according to the Poincare inequality.

If X is a Banach space, $L^p(0,T,X), 1\leq p<+\infty$ will be the linear space of measurable functions from the interval $(0,T)$ into X such that

$$
\int_0^T ||u(t)||_X^p \mathrm{d}t < \infty.
$$