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A FINITE ELEMENT METHOD FOR ELASTICITY INTERFACE PROBLEMS WITH LOCALLY MODIFIED TRIANGULATIONS

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Abstract. A finite element method for elasticity systems with discontinuities in the coefficients and the flux across an arbitrary interface is proposed in this paper. The method is based on a Cartesian mesh with local modifications to the mesh. The total degrees of the freedom of the finite element method remains the same as that of the Cartesian mesh. The local modifications lead to a quasi-uniform body-fitted mesh from the original Cartesian mesh. The standard finite element theory and implementation are applicable. Numerical examples that involve discontinuous material coefficients and non-homogeneous jump in the flux across the interface demonstrate the efficiency of the proposed method.

Key Words. elasticity interface problem, body-fitted mesh, Cartesian mesh, discontinuous coefficient, locally modified triangulation, finite element method, jump conditions

1. Introduction

In this paper, we propose a finite element method for plane elasticity problems with interfaces in which the physical parameters and solutions may be discontinuous across an arbitrary interface. Elasticity interface problems have wide applications in continuum mechanics, particularly for problems that involve stresses and strains, see for example, [4, 13, 20].

We first introduce the problem of our interest. Let $\mathbf{x} = (x, y)$ be a point in space and $\mathbf{u} = (u_1(x, y), u_2(x, y))$ be the displacement of a plate which is composed of different materials. The relation between strains and displacements of the plate is given by

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial y}, \quad \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right). \tag{1}$$

Assuming that the material is linearly elastic and isotropic; and that the displacements are small, we have the following relation between stresses and strains, or the constitutive relation from the Hooke's law,

$$\sigma_{ij} = \lambda\ (\nabla \cdot \mathbf{u})\ \delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u}), \quad i, j = 1, 2, \tag{2}$$

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where λ and μ are the Lamé coefficients, and

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}.$$

Let $\sigma = (\sigma_{ij})$ be the stress tensor, $\mathbf{f}(\mathbf{x}) = (f_1, f_2)$ be the applied body forces, then the stress tensor satisfies the following partial differential equations,

$$-\nabla \cdot \sigma = \mathbf{f}, \tag{3}$$

i.e.,

$$\begin{cases} -\dfrac{\partial \sigma_{11}}{\partial x} - \dfrac{\partial \sigma_{12}}{\partial y} = f_1, \\ -\dfrac{\partial \sigma_{21}}{\partial x} - \dfrac{\partial \sigma_{22}}{\partial y} = f_2. \end{cases} \tag{4}$$

From (2)-(4), we can re-write the above system as the system of plane elasticity equations of the following,

$$\begin{aligned} -\left\{(\lambda + 2\mu)\frac{\partial^2 u_1}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 u_2}{\partial x \partial y} + \mu\frac{\partial^2 u_1}{\partial y^2}\right\} &= f_1, \\ -\left\{(\lambda + 2\mu)\frac{\partial^2 u_2}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u_1}{\partial x \partial y} + \mu\frac{\partial^2 u_2}{\partial x^2}\right\} &= f_2. \end{aligned} \tag{5}$$

In the vector form, it is

$$-\mu \triangle \mathbf{u} - (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} = \mathbf{f}. \tag{6}$$

Note that, in practice, it is common to use the Young's modulus E and Poisson's ratio ν instead of the Lamé coefficients λ and μ in the expression (2). The relations between λ and μ, and E and ν, are given by

$$\mu = \frac{E}{2(1+\nu)}, \tag{7}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{(plane strain)}, \quad \lambda = \frac{\nu E}{1-\nu^2} \quad \text{(plane stress)}. \tag{8}$$

We want to obtain the numerical solution of the elasticity system that has an interface Γ in the solution domain. Across the interface Γ, the material coefficients may have finite jumps; so does the flux $\sigma\mathbf{n}$, see Fig. 1 for an illustration. Now the problem can be written as follows:

$$-\nabla \cdot \sigma = \mathbf{f} \quad \text{in } \Omega^+ \cup \Omega^- \tag{9}$$

$$[\mathbf{u}]_\Gamma = 0, \tag{10}$$

$$[\sigma\mathbf{n}]_\Gamma = \mathbf{q}, \tag{11}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0, \tag{12}$$

where $\mathbf{f} = (f_1, f_2)$, $\mathbf{q} = (q_1, q_2)$, $\mathbf{u}_0 = (u_{0_1}, u_{0_2})$ are known vector functions, and $\Gamma \in C^2$ is a closed interface between the subdomains Ω^+ and Ω^-. The jump $[\,\cdot\,]_\Gamma$ is defined as the difference of the limiting values from the outside of the interface to the inside, and $\mathbf{n}$ is the unit normal direction of the interface Γ pointing outward. We refer the reader to [15, 16] for more information of the elasticity problems.

It is always challenging to solve the interface problems. Several different approaches have been developed based on different formulations. A common and simple approach is to use a body-fitted mesh and a finite element method. This