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MEAN SQUARE CONVERGENCE OF STOCHASTIC θ-METHODS FOR NONLINEAR NEUTRAL STOCHASTIC DIFFERENTIAL DELAY EQUATIONS

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Abstract. This paper is devoted to the convergence analysis of stochastic θ-methods for nonlinear neutral stochastic differential delay equations (NSDDEs) in Itô sense. The basic idea is to reformulate the original problem eliminating the dependence on the differentiation of the solution in the past values, which leads to a stochastic differential algebraic system. Drift-implicit stochastic θ-methods are proposed for the coupled system. It is shown that the stochastic θ-methods are mean-square convergent with order $\frac{1}{2}$ for Lipschitz continuous coefficients of underlying NSDDEs. A nonlinear numerical example illustrates the theoretical results.

Key words. neutral stochastic differential delay equations, mean-square continuity, stochastic θ-methods, mean-square convergence, consistency

1. Introduction

Neutral delay differential equations (NDDEs) have found diverse applications in many fields such as control theory, oscillation theory, electrodynamics, biomathematics, and medical sciences. NDDEs arise in two formulations, explicit and implicit. Explicit NDDEs share the form of

$$x'(t) = F(t, x(t), x(t-\tau(t)), x'(t-\tau(t))), \quad t \in [t_0, T], \tag{1}$$

$$x(t) = \phi_0(t), x'(t) = \phi_1(t), \quad t \leq t_0, \tag{2}$$

where $\tau(t) \geq 0$. Implicit NDDEs share the form of

$$(x(t) - D(t, x(t), x(t-\tau(t)))' = F(t, x(t), x(t-\tau(t))), \quad t \in [t_0, T], \tag{3}$$

$$x(t) = \phi_0(t), x'(t) = \phi_1(t), \quad t \leq t_0, \tag{4}$$

which is also called "Hale's form". (3) can be formally rewritten as (1). However, under careful scrutiny, one may find that equation (3) is not necessarily equivalent to (1) even if D and τ are differentiable, noting that a non-differentiable function is probably a solution of (3). Therefore, the study of the explicit and implicit forms, and their available numerical methods differ. A stability analysis of both the exact solutions and the numerical approximations for explicit NDDEs has been presented in [3]. For the case of implicit NDDEs, Vermiglio and Torelli [23] investigated the stability of analytical solutions and the numerical approximations. The idea, which is to reformulate the original problem eliminating the dependence on the derivative of the solution in the last value, was presented both in [3] and [23]. There is an extensive literature on numerical schemes for NDDEs (see, for example, [2, 4, 8, 14]).

Many physical phenomena can be modelled by stochastic dynamical systems whose evolution in time is governed by random forces as well as intrinsic dependence of the state on a finite part of its past history. Such models may be identified as stochastic functional differential equations (SFDEs). The theory of SFDEs has been well developed and there is an extensive literature (see, for example, [18, 15]).

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The numerical methods on SFDEs have also been well established. Buckwar [6] discussed the strong convergence of drift-implicit one-step schemes to the solution of SFDEs. Fan, Liu and Cao [7] obtained sufficient conditions for the existence and uniqueness of solutions of stochastic pantograph equations and investigated the convergence of semi-implicit Euler method for the stochastic pantograph equations. There are other papers on numerical methods for SFDEs (see, for example, [5, 9, 16]). However, they are not on numerical methods for neutral SFDEs. A comprehensive introduction to numerical stochastic ordinary differential equations is given in the books by Allen [1], Kloeden and Platen [10], Kloeden et al [11], and Milstein and Tretyakov [17], and the surveys of Schurz [19], [20], [21] and Talay [22].

Motivated by chemical engineering systems and the theory of aeroelasticity, Kolmanovskii et al [12, 13] introduced a class of neutral stochastic functional differential equations. Mao [15] investigated existence and uniqueness, moment and pathwise estimates, exponential stability of neutral stochastic functional differential equation

$$d[x(t) - D(x_t)] = F(t, x_t)dt + G(t, x_t)dW(t). \tag{5}$$

and a special case of (5), that is, neutral stochastic differential delay equation

$$d[x(t) - D(x(t-\tau))] = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))dW(t). \tag{6}$$

To our best knowledge, no results on convergence of numerical methods for (5) and (6) have been presented in the literature. This paper is devoted to the convergence analysis of the stochastic θ-methods for nonlinear neutral stochastic differential delay equations (6). The basic idea is to transfer the system (6) into a stochastic ordinary differential system plus a functional recursion and then eliminate the dependence on the differentiation of the solution in the past values, which leads to a stochastic differential algebraic system. A drift-implicit stochastic θ-method is proposed for the coupled system. It is shown that these stochastic θ-methods are mean-square convergent with order $\frac{1}{2}$ under the usual smoothness assumptions. A nonlinear numerical example illustrates the theoretical results.

2. Neutral stochastic differential delay equations

Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^d$ and $\langle x, y\rangle$ be the Euclidean inner product of vectors $x, y \in \mathbb{R}^d$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$.

Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). $W(t) = (W_1(t), \ldots, W_m(t))^T$ is supposed to be a standard m-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with mutually independent coordinates W_i throughout the paper.

Furthermore, let $0 \leq t_0 < T < \infty$, $\mathcal{B}^d$ be the Borel σ-algebra and

$$F : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \quad G : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m} \text{ and } \ D : \mathbb{R}^d \to \mathbb{R}^d$$

be all Borel measurable real-valued functions. Consider the d-dimensional neutral stochastic differential delay equations (NSDDEs) in Itô-sense

$$d[x(t) - D(x(t-\tau))] = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))dW(t), \ t \in [t_0, T] \tag{7}$$

with initial data

$$x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0]. \tag{8}$$