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CONVERGENCE AND STABILITY OF THE SEMI-IMPLICIT EULER METHOD WITH VARIABLE STEPSIZE FOR A LINEAR STOCHASTIC PANTOGRAPH DIFFERENTIAL EQUATION

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Abstract. The paper deals with convergence and stability of the semi-implicit Euler method with variable stepsize for a linear stochastic pantograph differential equation(SPDE). It is proved that the semi-implicit Euler method with variable stepsize is convergent with strong order $p = \frac{1}{2}$. The conditions under which the method is mean square stability are determined and the numerical experiments are given.

Key Words. Stochastic pantograph differential equation, mean square stability, semi-implicit Euler method with variable stepsize.

1. Introduction

The importance of stochastic differential delay equations (SDDEs) derives from the fact that many of the phenomena witnessed around us do not have an immediate effect from the moment of their occurrence. A patient, for example, shows symptoms of an illness days (or even weeks) after the day in which he or she was infected. In general, we can find many "systems", in almost any area of science (medicine, physics, ecology, economics, etc.), for which the principle of causality, i.e., the future state of a system is independent of the past states and is determined solely by the present, does not apply. In order to incorporate this time lag (between the moment an action takes place and the moment its effect is observed) to our models, it is necessary to include an extra term which is called time delay. The SDDEs can be regarded as a generalization of stochastic differential equations (SDEs) and delay differential equations (DDEs). During the last few decades, many authors have studied SDDEs. some important results are given, for example, conditions which guarantee the existence and uniqueness of an analytical solution [13, 14, 15] and stability conditions for both exact solutions and numerical solutions, etc. [2, 6, 11, 16].

It is well known that in the deterministic situation there is a very special delay differential equation: the pantograph equation

$$\begin{aligned} y'(t) &= \bar{a}y(t) + \bar{b}y(qt), \quad 0 \leqslant t \leqslant t_f, \\ y(0) &= y_0. \end{aligned} \tag{1.1}$$

where $q \in (0, 1)$. It arises in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, quantum mechanics and electrodynamics. In particular, it is used by Ockendon and Taylor[17] to study how the electric current is collected by the pantograph of an electric locomotive,

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from where it gets its name. In [17] the coefficients $\bar{a}, \bar{b}$ of Eq.(1.1) are constants. If we take into account the estimation error for system parameters as well as the environmental noise, it is better to estimate parameters $\bar{a}, \bar{b}$ as point estimator plus an error. By the central limit theorem, the error may be described by a normally distributed random variable. Then, Eq.(1.1) becomes the differential form

$$\begin{aligned} dX(t) &= [aX(t) + bX(qt)]dt + [cX(t) + dX(qt)]dW(t), \quad t > 0, \\ X(0) &= x_0, \end{aligned} \tag{1.2}$$

which is a linear stochastic pantograph differential equation. In Eq.(1.2), $a, b, c, d \in \mathbb{R}$, $q \in (0, 1)$, $W(t)$ is a one-dimensional standard wiener process. The initial value x_0 is a real -valued random variable. The first term on the right hand side of Eq.(1.2) is usually called the *drift* function and the second term is called the *diffusion* function. The integral version of equation (1.2) is given by

$$\begin{aligned} X(t) = x_0 + \int_0^t [aX(s) + bX(qs)]ds \\ + \int_0^t [cX(s) + dX(qs)]dW(s), \end{aligned} \tag{1.3}$$

for $t > 0$. The second integral in Eq.(1.3) is a stochastic integral which is to be interpreted as the Itô sense [5].

The study for stochastic pantograph equation has just begun. Baker and Buckwar [3] give the necessary analytical theory for existence and uniqueness of a strong solution of the linear stochastic pantograph equation

$$\begin{aligned} dX(t) &= [aX(t) + bX(qt)]dt + [\sigma_1 + \sigma_2 X(t) + \sigma_3 X(qt)]dW(t), \\ X(0) &= X_0. \end{aligned} \tag{1.4}$$

They also prove that the numerical solution produced by the continuous θ-method converges to the true solution with order 1/2. Liu et al. [12] give stability conditions of the analytical solution of the nonlinear stochastic pantograph equation and provide results concerning convergence and stability of the semi-implicit Euler method with constant stepsize. Fan [4] give the sufficient conditions that guarantee the existence and uniqueness of a strong solution to the nonlinear stochastic pantograph equation and proved that the semi-implicit Euler method with constant stepsize applied to the nonlinear equation has strong order 1/2.

When the numerical method with a constant stepsize is applied to the pantograph equation, the most difficult problem is the limited computer memory as shown in [9, 10]. In this paper, we use the semi-implicit Euler method with variable stepsize for a scalar test equation (1.2) to avoid the storage problem and discuss the convergence and stability properties of the method. The other reason of applying a numerical method with a variable stepsize is that when using the numerical method with a constant stepsize to Eq. (1.2), the resulting difference equation is not of fixed order.

The paper is organized as follows. In Section 2, we will introduce some notations and recall some properties of its analytical solution. In Section 3, we will prove that the semi-implicit Euler method with a variable stepsize is convergent to the true solution with order $\frac{1}{2}$ and mean-square stability if $\theta \in \left(\frac{|a|+|b|}{2|a|}, 1\right]$. We will provide some numerical examples in Section 4.