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OPTIMAL ERROR ESTIMATES OF THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR WILLMORE FLOW OF GRAPHS ON CARTESIAN MESHES

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Abstract. In this paper, we analyze a local discontinuous Galerkin method for the willmore flow of graphs. We derive the optimal error estimates for this nonlinear equation in one-dimension and in multi-dimensions for Cartesian meshes using completely discontinuous piecewise polynomial space with degree $k \geq 1$.

Key Words. local discontinuous Galerkin method, Willmore flow of graphs, stability, error estimates

1. Introduction

In this paper, we consider the error estimates of the local discontinuous Galerkin (LDG) method [23] for the Willmore flow of graphs

$$u_t + Q\nabla \cdot \left(\frac{1}{Q} \left(\boldsymbol{I} - \frac{\nabla u \otimes \nabla u}{Q^2} \right) \nabla(QH) \right) - \frac{1}{2} Q\nabla \cdot \left(\frac{H^2}{Q} \nabla u \right) = 0, \tag{1.1}$$

where Q is the area element

$$Q = \sqrt{1 + |\nabla u|^2} \tag{1.2}$$

and H is the mean curvature of the domain boundary Γ

$$H = \nabla \cdot \left(\frac{\nabla u}{Q} \right). \tag{1.3}$$

In [23], we developed a LDG method for the for the Willmore flow of graphs and gave a rigorous proof for its energy stability. In this method the basis functions used are discontinuous in space. The LDG discretization also results in a high order accurate, extremely local, element based discretization. In particular, the LDG method is well suited for *hp*-adaptation, which consists of local mesh refinement and/or the adjustment of the polynomial order in individual elements. In this paper, we will present the optimal error analysis for the LDG method of the Willmore flow of graphs on Cartesian meshes. The analysis is made for the fully nonlinear case and the results are valid for all space dimension $d \leq 3$ and polynomial degree $k \geq 1$. We generalize the analysis to fully nonlinear case comparing with analysis for linear fourth order equation in [13]. We also obtain the optimal accuracy results comparing with the results for continuous linear finite element method in [12].

The DG method is a class of finite element methods, using discontinuous, piecewise polynomials as the solution and the test space. It was first designed as a

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method for solving hyperbolic conservation laws containing only first order spatial derivatives, e.g. Reed and Hill [17] for solving linear equations, and Cockburn et al. [5, 4, 3, 6] for solving nonlinear equations. It is difficult to apply the DG method directly to the equations with higher order derivatives. The LDG method is an extension of the DG method aimed at solving partial differential equations (PDEs) containing higher than first order spatial derivatives. The first LDG method was constructed by Cockburn and Shu in [7] for solving nonlinear convection diffusion equations containing second order spatial derivatives. Their work was motivated by the successful numerical experiments of Bassi and Rebay [1] for the compressible Navier-Stokes equations. The idea of the LDG method is to rewrite the equations with higher order derivatives into a first order system, then apply the DG method on the system. The design of the numerical fluxes is the key ingredient to ensure stability. The LDG techniques have been developed for convection diffusion equations (containing second derivatives) [7], nonlinear one-dimensional and two-dimensional KdV type equations [25, 22] and Cahn-Hilliard equations [20, 21]. Recently, there is a review paper on the LDG methods for high-order time-dependent partial differential equations [24]. More general information about DG methods for elliptic, parabolic and hyperbolic partial differential equations can be found in the three special journal issues devoted to the DG method [9, 10, 11], as well as in the recent books and lecture notes [15, 14, 18, 19].

The paper is organized as follows. In Section 2, we give some notations, definition and projections. In Section 3, we show LDG scheme for the Willmore flow of graphs and the main results in this paper. In section 4, we give some auxiliary results which is important for our analysis. In section 5, we present the proof of the error estimates. Concluding remarks are given in Section 6. Some of the more technical proofs of several lemmas are collected in Appendix A.

2. Notations, definitions and projections

We first introduce notations, definitions and projections to be used later in the paper. We define some projections and present certain interpolation and inverse properties for the finite element spaces that will be used in the error analysis.

2.1. Tessellation and function spaces. Let $\mathcal{T}_h$ denote a tessellation of Ω with shape-regular elements K. Let Γ denote the union of the boundary faces of elements $K \in \mathcal{T}_h$, i.e. $\Gamma = \cup_{K\in\mathcal{T}_h}\partial K$, and $\Gamma_0 = \Gamma\backslash\partial\Omega$.

In order to describe the flux functions we need to introduce some notations. Let e be a face shared by the "left" and "right" elements K_L and K_R (we refer to [25] and [24] for a proper definition of "left" and "right" in our context). Define the normal vectors ν_L and ν_R on e pointing exterior to K_L and K_R, respectively. If ψ is a function on K_L and K_R, but possibly discontinuous across e, let ψ_L denote $(\psi|_{K_L})|_e$ and ψ_R denote $(\psi|_{K_R})|_e$, the left and right trace, respectively.

Let $\mathcal{Q}^k(K)$ be the space of tensor product of polynomials of degree at most $k \geq 0$ on $K \in \mathcal{T}_h$ in each variable. The finite element spaces are denoted by

$$V_h = \Big\{\varphi \in L^2(\Omega): \quad \varphi|_K \in \mathcal{Q}^k(K), \quad \forall K \in \mathcal{T}_h\Big\},$$
$$\Sigma_h = \Big\{\boldsymbol{\eta} = (\eta_1, \cdots, \eta_d)^T \in (L^2(\Omega))^d: \quad \eta_l|_K \in \mathcal{Q}^k(K), \ \ l = 1\cdots d, \quad \forall K \in \mathcal{T}_h\Big\}.$$

For one-dimensional case, we have $\mathcal{Q}^k(K) = \mathcal{P}^k(K)$ which is the space of polynomials of degree at most $k \geq 0$ defined on K. Note that functions in V_h and Σ_h are allowed to have discontinuities across element interfaces. Here we only consider periodic boundary conditions. Notice that the assumption of periodic boundary