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IMMERSED FINITE ELEMENT METHODS FOR ELLIPTIC INTERFACE PROBLEMS WITH NON-HOMOGENEOUS JUMP CONDITIONS

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Abstract. This paper is to develop immersed finite element (IFE) functions for solving second order elliptic boundary value problems with discontinuous coefficients and non-homogeneous jump conditions. These IFE functions can be formed on meshes independent of interface. Numerical examples demonstrate that these IFE functions have the usual approximation capability expected from polynomials employed. The related IFE methods based on the Galerkin formulation can be considered as natural extensions of those IFE methods in the literature developed for homogeneous jump conditions, and they can optimally solve the interface problems with a nonhomogeneous flux jump condition.

Key Words. Key words: interface problems, immersed interface, finite element, nonhomogeneous jump conditions.

1. Introduction

In this paper, we consider the following typical elliptic interface problems:

$$-\nabla \cdot \big(\beta \nabla u\big) = f(x,y),\ \ (x,y) \in \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = g(x,y) \tag{1.2}$$

together with the jump conditions on the interface Γ:

$$[u]\,|_{\Gamma} \;=\; 0, \tag{1.3}$$

$$\left[\beta \frac{\partial u}{\partial \mathbf{n}}\right]|_{\Gamma} \;=\; Q(x,y). \tag{1.4}$$

Here, see the sketch in Figure 1, without loss of generality, we assume that $\Omega \subset I\!R^2$ is a rectangular domain, the interface Γ is a curve separating Ω into two sub-domains Ω^-, Ω^+ such that $\overline{\Omega} = \overline{\Omega^-} \cup \overline{\Omega^+} \cup \Gamma$, and the coefficient $\beta(x,y)$ is a piecewise constant function defined by

$$\beta(x,y) = \begin{cases} \beta^-, & (x,y) \in \Omega^-, \\ \beta^+, & (x,y) \in \Omega^+. \end{cases}$$

Interface problem (1.1) - (1.4) appears in many applications. For example, the electric potential u satisfies jump conditions (1.3) and (1.4) on the interface between two isotropic media if the surface charge density Q on Γ is not zero [10]. Another example is the modeling of water flow in a domain consisting of two stratified porous media with a source at the interface between the media [35].

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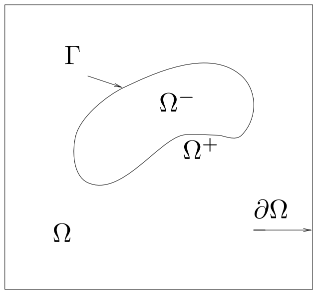

FIGURE 1. A sketch of the domain for the interface problem.

The interface problem (1.1) - (1.4) can be solved by conventional numerical methods, including both finite difference (FD) methods, see [17, 37] and references therein, and finite element (FE) methods, see [3, 6, 9] and references therein, provided that the computational meshes are body-fitting. A body-fitting mesh, see the illustration in Figure 2, is constructed according to the interface such that each element/cell in this mesh is essentially on one side of the interface. Physically, this means each element/cell in a body-fitting mesh is essentially occupied by one of the materials forming the simulation domain of the interface problem.

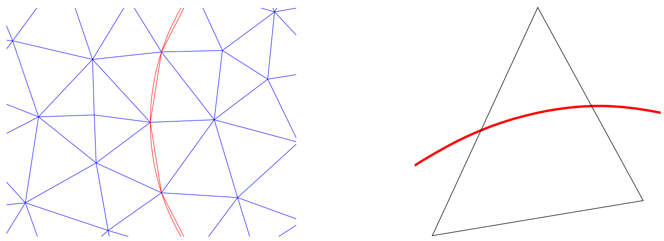

FIGURE 2. The plot on the left shows how elements are placed along an interface in a standard FE method. An element not allowed in a standard FE method is illustrated by the plot on the right.

For a non-trivial interface Γ, it is usually not possible to solve the interface problem on a structured mesh satisfactorily. On the other hand, there are applications, such as particle-in-cell simulation of plasma driven by the electric field in a Micro-Ion Thrusters [39, 40], in which it is preferable to solve the interface problem on a structured Cartesian mesh. Therefore, many efforts have been made for developing interface problem solvers that can use meshes independent of interface. In finite difference/volume formulation, we note the Cartesian grid methods [36], embedded boundary methods [18], immersed interface method [12, 20, 26, 27, 29], cut-cell methods [19, 21], matched interface and boundary methods [44, 45], etc.. In finite element formulation, Babuška et al. [4, 5] developed the generalized finite element method. Their basic idea is to form the local basis functions in an element by solving the interface problem locally. The local basis functions in their method can capture important features of the exact solution and they can even be non-polynomials. The recently developed immersed finite element (IFE) methods [1, 2, 8, 11, 13, 15, 16, 24, 25, 28, 30, 31, 32, 33, 34, 38, 41] also fall into this framework. The IFEs are developed such that their mesh can be independent of the interface, but the local basis functions are constructed according to the interface jump conditions; hence,