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# MIXED FEM OF HIGHER ORDER FOR CONTACT PROBLEMS WITH FRICTION

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**Abstract.** This paper presents a mixed variational formulation and its discretization by finite elements of higher-order for the Signorini problem with Tresca friction. To guarantee the unique existence of the solution to the discrete mixed problem, a discrete inf-sup condition is proved. Moreover, a solution scheme based on the dual formulation of the problem is proposed. Numerical results confirm the theoretical findings.

**Key words.** *hp*-FEM, mixed method, contact problems, Signorini problem, friction

## 1. Introduction

This paper deals with finite element methods of higher-order for the Signorini problem with Tresca friction, which plays an important role in mechanical engineering [14, 15, 24]. The discretization approach is based on a mixed variational formulation. For lower-order finite elements, this approach was introduced by Haslinger et al. in [16, 19, 21]. In this paper, we extend it to higher-order finite elements. The approach relies on a saddle point formulation where the geometrical contact condition and the frictional condition are captured by Lagrange multipliers. The constraints for the Lagrange multipliers are sign conditions and box constraints and are, therefore, simpler than the original contact conditions. However, the Lagrange multipliers are additional variables which also have to be discretized. In mixed variational formulations, unique existence of the discrete saddle point usually follows from an inf-sup condition associated to the discretization spaces. Its verification is often a crucial point. For lower-order finite elements, the inf-sup condition is proved in the above mentioned references. In this work, we prove the inf-sup condition for higher-order finite elements for the Signorini problem with Tresca friction. We use approximation results for the $p$-method of finite elements, and some inverse estimates for higher-order polynomials, [2, 11]. The key is to use a discretization of the Lagrange multipliers on boundary meshes with a larger mesh size than that of the primal variable and, moreover, different polynomial degrees for the primal variable and Lagrange multipliers.

In general, higher-order discretization schemes for contact problems are rarely studied in literature, especially for mixed variational formulation. For discretization techniques based on a primal, non-mixed formulation, we refer to [26, 27].

This paper is organized as follows: To motivate the subject, to show the analytical background behind and, in particular, to introduce the discrete inf-sup condition, we briefly summarize the main arguments of convex analysis for the derivation of a mixed variational formulation in Section 2. If necessary, some of the proofs are given in the appendix. In Section 3, we apply the abstract framework to obtain a mixed variational formulation for the Signorini problem with Tresca

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friction and to introduce a higher-order finite element discretization. In Section 4, we consider some simplifications of the Signorini problem and also assert them to the abstract framework of Section 2. The main part of this work, the derivation of the inf-sup condition for higher-order finite elements, is proposed in Section 5.
The second focus of this work is to present a solution scheme to solve the discrete mixed variational formulation. The scheme is based on a dual variational formulation leading to a minimization problem in terms of the Lagrange multipliers. It follows the same line as in the approach presented in [18, 20, 17]. In Section 6, we extend it to the higher-order approach. Furthermore, we discuss an extension of the solution scheme to time-dependent problems in Section 7. Numerical results confirming the theoretical findings are presented in Section 8.

## 2. General remarks on mixed variational formulations

Frictional contact problems can be captured by the minimization problem

$$(H+j)(u) = \min_{v\in K}(H+j)(v). \tag{1}$$

Here, $K$ is a subset of a reflexive Banach space $V$ and $H, j : V \to \mathbb{R}$. The special choice for $V$, $H$ and $j$ in the context of contact problems with friction will become clear in Section 3, below.

**Theorem 1.** *Let $K$ be convex.*

*(i) If $K$ is closed and $H+j$ is weakly lower semicontinuous and coercive, then there exists a minimizer $u \in K$ of (1).*
*(ii) If $H+j$ is strictly convex, (1) admits at most one minimizer.*
*(iii) Let $H$ be Fréchet differentiable in $u \in K$ with the Fréchet derivative $H'(u) \in V'$. If $u$ is a minimizer of (1) and $j$ is convex, then*

$$\langle H'(u), v-u\rangle + j(v) - j(u) \geq 0 \tag{2}$$

*for all $v \in K$. If $H$ is convex and (2) holds, then $u$ is a minimizer of (1).*

The assertions of Theorem 1 are well-known. For (i)-(ii), we refer to [24, Prop. 3.1, p.33] and [10, Ch. II, Prop. 1.2, p.35]. The proof of the assertion (iii) is given in the appendix.
To derive a mixed variational formulation, we resolve the condition $v \in K$ and the functional $j$ by using Lagrange multipliers. To this end, let $\Phi_i : V \times \Lambda_i \to \mathbb{R}$, $i = 0, 1$, fulfill

$$\sup_{\mu_0\in\Lambda_0} \Phi_0(v,\mu_0) = \begin{cases} 0, & v \in K \\ \infty, & v \notin K \end{cases} \tag{3}$$

and

$$j(v) = \sup_{\mu_1\in\Lambda_1} \Phi_1(v,\mu_1) \tag{4}$$

for all $v \in V$ with $\Lambda_i \subset U_i'$ and reflexive Banach spaces $U_i'$. Obviously, it holds

$$(H+j)(u) = \inf_{v\in V} \sup_{\mu_0\in\Lambda_0,\mu_1\in\Lambda_1} \mathcal{L}(v,\mu_0,\mu_1)$$

with the Lagrange functional $\mathcal{L}(v,\mu_0,\mu_1) := H(v) + \Phi_0(v,\mu_0) + \Phi_1(v,\mu_1)$. Therefore, $u$ is a minimizer of (1), whenever the triple $(u,\lambda_0,\lambda_1) \in V \times \Lambda_0 \times \Lambda_1$ is a saddle point,

$$\mathcal{L}(u,\lambda_0,\lambda_1) = \inf_{v\in V} \sup_{\mu_0\in\Lambda_0,\mu_1\in\Lambda_1} \mathcal{L}(v,\mu_0,\mu_1). \tag{5}$$

Defining $\Phi_{i,\mu_i}(v) := \Phi_i(v,\mu_i)$ and $\Phi_{i,v}(\mu_i) := \Phi_i(v,\mu_i)$ and applying Theorem 1, we immediately obtain