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A NUMERICAL APPROACH FOR SOLVING A CLASS OF SINGULAR BOUNDARY VALUE PROBLEMS ARISING IN PHYSIOLOGY

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Abstract. In this paper, two numerical schemes for finding approximate solutions of singular two-point boundary value problems arising in physiology are presented. While the main ingredient of both approaches is the employment of cubic B-splines, the obstacle of singularity has to be removed first. In the first approach, L'Hopital's rule is used to remove the singularity due to the boundary condition (BC) $y'(0) = 0$. In the second approach, the economized Chebyshev polynomial is implemented in the vicinity of the singular point due to the BC $y(0) = A$, where A is a constant. Numerical examples are presented to demonstrate the applicability and efficiency of the methods on one hand and to confirm the second order convergence on the other hand.

Key words. Boundary value problems; Chebyshev polynomial; B-spline; Singularities

1. Introduction

Many numerical treatments for singular boundary value problems have emerged in recent years. To mention few, Pandey and Singh [14] applied a finite difference method on a uniform mesh for a class of singular boundary value problem (BVP), Kanth and Bhattacharya [12] employed B-spline functions after reducing the nonlinear problem into a sequence of linear problems by using quesilinearization techniques, and then modifying the resulting sets of differential equations around the singular point. For a homogenous and linear singular BVP, Kadalbajoo and Aggarwal [11] started by finding a series solution in the vicinity of the singularity and then applying a cubic spline method for the remaining part of the interval. The reader may also see [6] and [7] for extra readings.

In this paper, we will consider a more general nonlinear singular BVP of the form

$$(p(x)y')' = p(x)f(x,y), \quad x \in (0,1] \tag{1}$$

with boundary conditions (BC)

$$y'(0) = 0, \quad \alpha y(1) + \beta y'(1) = \gamma \tag{2}$$

or

$$y(0) = A, \quad \alpha y(1) + \beta y'(1) = \gamma \tag{3}$$

where

$$p(x) = x^b g(x), \ x \in [0,1] \tag{4}$$

here $\alpha > 0$, $\beta \geq 0$ and A and γ are finite constants. Also, the following restrictions are imposed on $p(x)$ and $f(x,y)$.

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(I) $p(x) > 0$ on $[0,1]$, $p(x) \in C^1(0,1]$, and $1/g(x)$ is analytic in $\{z \text{ s.t. } |z| < r\}$ for some $r > 1$.

(II) $f(x,y) \in [0,1] \times R$, is continuous, $\partial f/\partial y$ exists, continuos, and nonnegative for all $(x,y) \in [0,1] \times R$.

The existence-uniqueness of (1) has been established for BCs $y(0) = A$ and $y(1) = B$ with $0 \le b < 1$, and BCs $y'(0) = 0$ and $y(1) = B$ with $b \ge 0$, provided that xp'/p is analytic in $\{z \text{ s.t. } |z| < r\}$ for some $r > 1$ [15–17].

The BVP (1) with BC (3) arises in the study of tumor growth problems [1–3, 10] where $b = 0, 1, 2$, $g(x) = 1$, and $f(x,y)$ is either linear or nonlinear of the form

$$f(x,y) = \frac{\theta y}{y+\kappa}, \qquad \theta, \kappa > 0. \tag{5}$$

and also arises in the study of a steady-state oxygen diffusion in a cell with Michaleis-Menten uptake kinetics when $b = 2$ and $g(x) = 1$ [13].

Also, a similar problem arises in the study of the distribution of heat sources in the human head [8, 9] where $b = 2$, $g(x) = 1$, and

$$f(x,y) = -\delta e^{-\theta y}, \qquad \theta, \delta > 0. \tag{6}$$

In this paper, we propose two numerical schemes to find approximate solutions for (1) with BC (2), and for (1) with BC (3) for a wider range of b and for the case where $f(x,y)$ is nonlinear. The paper is outlined as follows: In section 2, we describe the two methods for the two sets of boundary conditions (2) and (3). In section 3, two applications to physiology are presented and the results are compared to those obtained by [14], also the second-order of convergence will be verified as was established by Ahlberg and Ito [4]. Some concluding remarks are summarized in section 4.

2. Description of the method

Substituting the value of $p(x)$ in (4) into (1), and algebraic manipulations yields

$$y'' + \left(\frac{b}{x} + \frac{g'(x)}{g(x)}\right) y' = f(x,y) \tag{7}$$

Each set of boundary conditions (2) and (3) will be treated separately.

2.1. L'Hopital's rule and cubic B-splines.

In this section, we discuss a method for solving BVP (1) with BCs (2), which is analogous to, but more general than, that introduced in [5].

To overcome the singularity at $x = 0$, we apply L'Hopital's rule as x approaches zero to the term $\frac{b}{x}y'$ in (7). So we obtain the boundary value problem

$$y'' + \left(\frac{\mu}{x} + \eta\frac{g'(x)}{g(x)}\right) y' = F(x,y) \tag{8}$$

where

$$F(x,y) = \begin{cases} f(x,y), & x \neq 0 \\ \frac{1}{b+1} f(0,y), & x = 0 \end{cases} \tag{9}$$

and

$$\begin{cases} \mu = b \text{ and } \eta = 1, & x \neq 0 \\ \mu = 0 \text{ and } \eta = \frac{1}{b+1}, & x = 0 \end{cases} \tag{10}$$