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ELEMENT-BY-ELEMENT POST-PROCESSING OF DISCONTINUOUS GALERKIN METHODS FOR NAGHDI ARCHES

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Abstract. In this paper, we consider discontinuous Galerkin approximations to the solution of Naghdi arches and show how to post-process them in an element-by-element fashion to obtain a far better approximation. Indeed, we prove that, if polynomials of degree k are used, the post-processed approximation converges with order $2k+1$ in the L^2-norm *throughout the domain*. This has to be contrasted with the fact that before post-processing, the approximation converges with order $k+1$ only. Moreover, we show that this superconvergence property does not deteriorate as the thickness of the arch becomes extremely small. Numerical experiments verifying the above-mentioned theoretical results are displayed.

Key words. Post-processing, superconvergence, discontinuous Galerkin methods, Naghdi arches

1. Introduction

In [5], a family of discontinuous Galerkin (DG) methods for a Naghdi-type arch model was introduced as a step towards the difficult goal of devising locking-free DG methods for shells. They have proved that the approximation converges with order $k+1$ when polynomials of degree k are used. In this paper, we construct an element-by-element post-processing that converges remarkably faster.

This post-processing is based on the fact that a superconvergence phenomenon takes place at the nodes of the mesh. Indeed, the numerical traces of the DG method converge to the nodal values of the exact solution with order $2k+1$ when polynomials of degree k are used to compute the DG approximation, see [5]. The main goal of this paper is to exploit this phenomenon to post-process the DG solution element-by-element and obtain a better solution which superconverges to the exact solution with order $2k+1$ in the L^2-norm throughout the domain rather than at merely some isolated points of the mesh.

A similar superconvergent post-processing result has been proved for DG methods for convection-diffusion problems in [3]. Based on the superconvergence result proved therein, Cockburn and Ichikawa [7] devised a post-processing for the approximation of linear functionals which is superconvergent of order $4k+1$. In [2] Celiker and Cockburn designed a post-processing for DG methods for Timoshenko beams which is superconvergent of order $2k+1$ in the L^∞-norm throughout the computational domain. This result was based on the numerical observation that the numerical traces of the DG approximation for Timoshenko beams are also superconvergent of order $2k+1$ at the nodes of the mesh. Shortly later, the superconvergence of the numerical traces was put on a firm mathematical ground in [4].

As we will describe below, the Timoshenko beam model can be viewed as a special case of the Naghdi arch model where the beam is considered as an arch with zero curvature. The post-processing we display in this paper is thus inspired

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by the one introduced in [2]. Despite this close similarity, the coupling of some of the unknowns in the Naghdi arch model renders both the post-processing and its error analysis more involved. This is especially the case for the latter because it requires the analysis of a linear system of initial value problems whose solution is approximated by using approximate data. This is the main reason why we prove an L^2-error estimate for the post-processed approximation unlike the L^∞-error estimate for the Timoshenko beam post-processing. Notwithstanding, it is possible to prove an L^∞-error estimate at the expense of requiring high order regularity, following, for example, [11, 17].

Next, we describe the Naghdi arch model . A dimensionless form of this model can be written as a system of first order differential equations:

$$w' + \theta + \kappa u = d^2 T, \tag{1a}$$

$$u' - \kappa w = d^2 N, \tag{1b}$$

$$\theta' + \kappa(u' - \kappa w) = M, \tag{1c}$$

$$M' = T, \tag{1d}$$

$$N' + (\kappa M)' - \kappa T = p, \tag{1e}$$

$$T' + \kappa^2 M + \kappa N = q, \tag{1f}$$

defined on $\Omega = (0, 1)$. For the simplicity of our notation we have assumed that the model is non-dimensionalized in a way that all the material properties including the Young's modulus, shear modulus, moment of inertia, and the length of the arch are scaled to be equal to one. However, all the results in this paper can be generalized to the case in which they are non-constant functions. The small parameter $d > 0$ represents the dimensionless thickness of the arch. The function κ is x-dependent, and $\kappa(x)$ is the curvature of the middle curve of the arch at the point of coordinate x. When κ is constantly valued, the arch is circular. A straight beam could be viewed as a special arch with $\kappa \equiv 0$, in which case (1) decouples to the Timoshenko beam bending model. The functions p and q are the tangential and transverse resultant loads, respectively. Similarly, a displacement vector of a point of the middle curve is decomposed to its tangent component u and normal component w. The remaining unknowns are the rotation of the normal fibers, θ, the bending moment, M, the scaled membrane stress, N, and the scaled shear stress, T. In Figure 1 we display some of the characteristics of a typical arch. The parametrization is indicated by

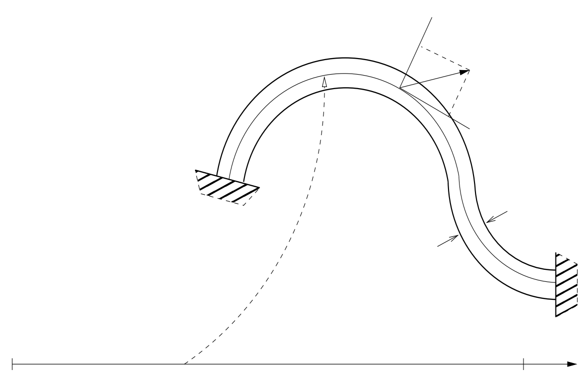

FIGURE 1. Cross section of an arch clamped at both ends, and arc length parametrization of its middle curve.