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A SEMI-IMPLICIT BINARY LEVEL SET METHOD FOR SOURCE RECONSTRUCTION PROBLEMS

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Abstract. The aim of this paper is to investigate the application of a semi-implicit additive operator splitting scheme based binary level set method to source reconstruction problems. We reformulate the original model to be a new constrained optimization problem under the binary level set framework and solve it by the augmented Lagrangian method. Then we propose an efficient gradient-type algorithm based on the additive operator splitting scheme. The proposed algorithm can create new holes during the evolution. Topological changes can be handled automatically and complex geometry can be recovered under a certain amount of noise in the observation data. Numerical examples are presented to show the effectiveness and efficiency of our method.

Key Words. source reconstruction, binary level set method, augmented Lagrangian method, additive operator splitting.

1. Introduction

Let $U \subset \mathbb{R}^2$ be an open bounded domain and $\mathcal{K}_{ad}$ be a class of admissible compact subsets of U. Consider the following nonlinear output-least-squares problem

$$\min_{\Omega \in \mathcal{K}_{ad}} F(\Omega), \quad F(\Omega) = \frac{1}{2} \int_M |u(\Omega) - u^*|^2 \mathrm{d}x, \tag{1}$$

where the observation data u^* defined on a set $M \subset U$ or $M \subset \partial U$ typically contains noise. The relationship between u and Ω is given by the elliptic equation

$$-\Delta u = \chi_\Omega \ \text{ in } U \tag{2}$$

subject to homogeneous Dirichlet boundary conditions on ∂U, where χ_Ω is the characteristic function of Ω, i.e.,

$$\chi_\Omega = \begin{cases} 1 & \text{in } \Omega, \\ 0 & \text{in } U \setminus \Omega. \end{cases}$$

Given the noisy observation u^* of u, the aim is to find the optimal shape of Ω which satisfies the state equation (2) and fits the observation data best.

The source reconstruction problem (1) belongs to shape recovery (cf. [3, 4, 5, 6, 7, 13]), which is a very popular and challenging field of inverse problems. Other examples of shape reconstruction and identification include diffraction screen (cf. [19, 25]), identification of cavities (cf. [1, 3]), electrical impedance tomography (cf. [8, 10, 14]), etc. For solving such problems, one needs to find a mechanism to represent the shape and follow its evolution. Many effective and efficient methods

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have been motivated. Hettlich and Rundell [13] solved an inverse source problem from measurements of the Neumann boundary values of u on ∂U. But their numerical methods based on boundary variation and shape derivative calculation [27] could only reconstruct a regular annular obstacle and failed to recover non-simply connected shapes. In [12], a phase-field method based on the Ginzburg-Landau regularization was employed for the solution of an inverse conductivity problem. Multi-connected inclusions can be recovered by the implicit representation of the shape. However, thousands of iterations were required for solving the gradient descent flow by the explicit Euler time stepping. The level set method originally proposed by Osher and Sethian [24] for interface evolution and tracking has been applied in many fields [22]. The interface is represented implicitly by the zero level set of a Lipschitz continuous function. Effective difference schemes for the Hamilton-Jacobi equation can be implemented on fixed grids. Moreover, certain types of shape and topological changes, such as merging, splitting and developing sharp corners can be handled automatically. The combination of level set methods with shape sensitivity analysis [27] has been widely applied to many shape recovery problems (see, e.g., [3, 5, 8, 9, 10, 14, 25]) and optimal shape design (see, e.g., [2, 23]). We refer to see the surveys [4, 7, 28] and the references therein. For such problems, the interface is often evolved with a given velocity obtained by calculating the shape gradient of the objective functional.

However, as pointed out in [2, 6], the shape gradient based level set method can not create new holes, which may cause the algorithm to get stuck at local shapes with fewer holes than the optimal geometry. The reconstructed results therefore largely depend on the initial guess. The topology derivative [26] proposed for hole nucleation has been combined with shape derivative in level set methods to solve inverse source problems in [6], where numerical examples demonstrated that the new algorithm can recover certain shapes that the shape gradient based level set method fails to. But the topology sensitivity analysis is generally quite complicated. Newton-type level set methods, such as the Gauss-Newton method (cf. [25]) and the Levenberg-Marquardt approach (cf. [5]), can have a decrease in the number of iterations compared with gradient-type methods, but they require the inversion of a large dense sensitivity matrix in each iteration, which is computationally slow.

Explicit schemes for the level set based gradient flow typically suffer from the Courant-Friedrichs-Lewy (CFL) stability condition [22], which means that an iterative algorithm requires rather many iterations to reach a stationary state. In order to accelerate convergence by reducing the number of iterations, the semi-implicit additive operator splitting (AOS) scheme [17, 32] was incorporated into the traditional level set method [18] and the piecewise constant level set approach [31] for efficiently solving structural topology optimization. This scheme avoids the CFL restriction and is unconditionally stable. It treats all the spatial variables in a symmetrical way. At each iteration, the computational effort for solving the tri-diagonal linear systems by the fast Thomas algorithm is moderate. It has locally second order of accuracy and globally first order of accuracy.

The level set approach of binary type was firstly proposed in [16] with application to image segmentation and later employed to the solution of elliptic coefficient inverse problems [20] and shape optimization [34]. The binary level set method (BLSM) is closely related to the phase-field method as pointed out in [16]. Similar as the multiple level set method [8, 28], the BLSM also requires N level set functions to represent up to 2^N subregions. In the BLSM, however, interfaces are identified implicitly by the discontinuities of the binary level set functions (BLSFs) taking