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NUMERICAL APPROXIMATION OF OPTION PRICING MODEL UNDER JUMP DIFFUSION USING THE LAPLACE TRANSFORMATION METHOD

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Abstract. We propose a **LT** (*Laplace transformation*) method for solving the PIDE (*partial integro-differential equation*) arising from the financial mathematics. An option model under a jump-diffusion process is given by a PIDE, whose non-local integral term requires huge computational costs. In this work, the PIDE is transformed into a set of complex-valued elliptic problems by taking the Laplace transformation in time variable. Only a small number of Laplace transformed equations are then solved on a suitable choice of contour. Then the time-domain solution can be obtained by taking the Laplace inversion based on the chosen contour. Especially a splitting method is proposed to solve the PIDE, and its solvability and convergence are proved. Numerical results are shown to confirm the efficiency of the proposed method and the parallelizable property.

Key Words. Laplace inversion, Option, Derivative, Jump-diffusion

1. Introduction

The Black-Scholes formula [3], introduced by Black and Scholes in 1973, has been adopted as a standard framework for option pricing, particularly due to its closed form solution. However, the difficulty in capturing a large or sudden movement of an underlying asset has been pointed out as a major drawback of the Black-Scholes formula. Practitioners and theorists have tried to extend the model of the underlying to tackle this phenomena, and among them the implementation of jump-diffusion processes has become one of the most popular tools. In the pioneering work of Merton [23], he modeled the underlying assets using a Brownian motion with drift having jumps arriving accordingly as a compound Poisson process. In [16], Kou tried to explain high peaks and heavy tails in asset return distributions incorporating with the volatility smile by proposing a double exponential jump diffusion model. These jump-diffusion processes are considered as specific examples of Lévy processes with stationary independent increments, and have been applied intensively to improve option modeling. For further details, for instance, readers are referred to the book by Cont and Tankov [6] and the references therein.

To evaluate the value of the option modeled by jump-diffusion process, one needs to solve a PIDE (*partial integro-differential equation*) of parabolic type that contains both partial differential operators and a non-local integral term. Several attempts have been tried to reduce the expensive computational cost of solving the PIDE. Most of such attempts may be roughly classified into two types. One is to try to improve the efficiency in the computation of the non-local integral term, and the

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other is try to reduce the number of time steps. In particular, the discretization of the non-local integral term generates dense matrices that are very expensive to deal with: iterative methods or other splitting techniques should be employed instead of any direct matrix inversion schemes. For example, an implicit-explicit method was developed by Zhang [32], and an ADI (*Alternative Direction Implicit*) method was applied by Andersen and Andersen [2]. A fixed point iteration scheme was introduced by d' Halluin *et al.* [8]. As an effort to reduce the number of time steps, Almendral and Oosterlee [1] applied a second order backward difference formula (BDF2) and Feng and Linetsky [9] used a high order extrapolation method.

In spite of the popularity of time marching methods, which all the above mentioned works employed, they require usually as many time steps as spatial meshes in order to balance the errors arising from the spatial and time discretizations. The schemes in [1, 9] reduce the number of time steps considerably compared to previous works, but they are still of polynomial convergence in time.

In the present paper, we propose a new approach for a parabolic type PIDE based on the **LT** (*Laplace transformation*) method. As proven in [26], the method is of exponential convergence in time, and in addition it can be easily parallelized. In the current paper, the method is coupled with a finite element method to solve the Laplace transformed complex elliptic equations. The solvability of the Laplace transformed equations is proved, and numerical experiments are performed to show the efficiency of the proposed method. The numerical results show an exponential order of convergence in time, and a second order in space. The **LT** method has been already applied to the Black-Scholes equation in [17]. Moreover, in [17], a precise absorbing boundary condition is derived and the solvability of the set of complex-valued elliptic problems that are the Laplace transforms of the Black-Scholes equation. Related with **LT** method there are other approaches; for instance, see [12, 10, 11, 20], and so on. Also, high-dimensional parabolic problems can be solved using sparse grids [13, 18, 19, 25]. Application of our **LT** method using sparse grids to option pricing will also be interesting. Other approaches in the fast time-stepping methods can be found in [29, 22, 21]; any of these methods can be also chosen in the **LT** method that is to be developed in this paper.

The rest of the paper is organized as follows. In the following subsection, a brief explanation about the mathematical formulation of the model under jump diffusion is described in terms of PIDE. §2 introduces the **LT** method and provides a convergence theorem with some remarks. In §3, we describe the Laplace transformed equation and prove its solvability. The finite element method for solving the complex elliptic equations and the technique to accelerate the numerical scheme are described in §4. §5 shows numerical results to confirm the convergence and efficiency results of the proposed method.

1.1. The parabolic integro-differential equation. Let S denote the price of an underlying asset. Following Merton [23], the underlying asset that is governed by a Brownian motion with drift having jumps arriving accordingly as a compound Poisson process is assumed to satisfy the following stochastic differential equation:

$$\mathrm{d}S = \nu S\,\mathrm{d}\tau + \sigma S\,\mathrm{d}W + (\eta - 1)S\,\mathrm{d}q, \tag{1.1}$$

where the terms $\mathrm{d}W$ and $\mathrm{d}q$ represent the increment of a Brownian motion and a Poisson process, respectively. In (1.1), ν and σ are the drift rate and the volatility of the Brownian part. The Poisson process $\mathrm{d}q$ is defined by

$$\mathrm{d}q = \begin{cases} 0 & \text{with probability } 1 - \lambda\,\mathrm{d}\tau, \\ 1 & \text{with probability } \lambda\,\mathrm{d}\tau, \end{cases}$$