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NUMERICAL SOLUTIONS OF NONLINEAR PARABOLIC PROBLEMS BY MONOTONE JACOBI AND GAUSS–SEIDEL METHODS

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Abstract. This paper is concerned with solving nonlinear monotone difference schemes of the parabolic type. The monotone Jacobi and monotone Gauss–Seidel methods are constructed. Convergence rates of the methods are compared and estimated. The proposed methods are applied to solving nonlinear singularly perturbed parabolic problems. Uniform convergence of the monotone methods is proved. Numerical experiments complement the theoretical results.

Key Words. nonlinear parabolic problem, monotone iterative method, singularly perturbed problem, uniform convergence.

1. Introduction

Many reaction-diffusion-convection-type problems in the chemical, physical and engineering sciences are described by nonlinear parabolic equations. The parabolic problem under consideration is in the form

$$\frac{\partial u}{\partial t} - Lu + f(x,t,u) = 0, \quad (x,t) \in \omega \times (0,T], \tag{1}$$

$$u(x,t) = g(x,t), \quad (x,t) \in \partial\omega \times (0,T], \quad u(x,0) = u^0(x), \quad x \in \overline{\omega},$$

where ω is a connected bounded domain in $\mathbb{R}^\kappa$ ($\kappa = 1, 2, \ldots$) with boundary $\partial\omega$. Lu is given by

$$Lu = \sum_{\nu=1}^{\kappa} \frac{\partial}{\partial x_\nu}\left(k_\nu(x,t)\frac{\partial u}{\partial x_\nu}\right) + \sum_{\nu=1}^{\kappa} v_\nu(x,t)\frac{\partial u}{\partial x_\nu},$$

where the coefficients of the differential operator are smooth and $k_\nu > 0$, $\nu = 1, \ldots, \kappa$, in $\overline{\omega}$. It is also assumed that the functions f and g are smooth in their respective domains.

In the study of numerical methods for nonlinear parabolic problems, the two major points to be developed are: i) constructing convergent nonlinear difference schemes and ii) computing solutions of nonlinear discrete problems. A major point about the nonlinear difference schemes is to obtain reliable and efficient computational methods for computing the solution. The reliability of iterative techniques for solving nonlinear difference schemes can be essentially improved by using componentwise monotone globally convergent iterations. Such methods can be controlled every time. A fruitful method for the treatment of these nonlinear schemes is the method of upper and lower solutions and its associated monotone iterations [7].

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Since an initial iteration in the monotone iterative method is either an upper or lower solution, which can be constructed directly from the difference equation without any knowledge of the exact solution, this method simplifies the search for the initial iteration as is often required in the Newton method. In the context of solving systems of nonlinear equations, the monotone iterative method belongs to the class of methods based on convergence under partial ordering (see Chapter 13 in [7] for details).

The purpose of this paper is to extend the monotone iterative method from [4] to monotone relaxation methods of Jacobi- and Gauss–Seidel type iterations for solving nonlinear monotone difference schemes in the canonical form and to apply the monotone methods to nonlinear singularly perturbed equations of the parabolic type. Convergence rates of these relaxation methods are compared and estimated.

The structure of the paper is as follows. In Section 2, we present the nonlinear monotone difference scheme in the canonical form and formulate the maximum principle. In Section 3, we construct the monotone Jacobi and monotone Gauss–Seidel methods, prove monotone convergence of the methods and compare their convergence rates. Section 4 is devoted to estimation of convergence rates of the monotone methods. In the final Section 5, the monotone methods are applied to solving nonlinear singularly perturbed parabolic problems. We prove that on layer-adapted meshes the monotone methods converge uniformly in a perturbation parameter. Numerical experiments complement the theoretical results.

2. The nonlinear difference scheme

On $\overline{\omega}$ and $[0,T]$, we introduce meshes $\overline{\omega}^h$ and $\overline{\omega}^\tau$, respectively. For simplicity, we assume that the mesh $\overline{\omega}^\tau$ is uniform with the time step τ. For a mesh function $U(p,t)$, $(p,t) \in \overline{\omega}^h \times \overline{\omega}^\tau$, consider the nonlinear implicit difference scheme in the canonical form [9]

$$\mathcal{L}U(p,t) + f(p,t,U) - \tau^{-1}U(p,t-\tau) = 0, \quad (p,t) \in \omega^h \times (\overline{\omega}^\tau \setminus 0), \tag{2}$$
$$U(p,0) = u^0(p), \quad p \in \overline{\omega}^h, \quad U(p,t) = g(p,t), \quad (p,t) \in \partial\omega^h \times (\overline{\omega}^\tau \setminus 0),$$

where $\partial\omega^h$ is the boundary of $\overline{\omega}^h$, and the difference operator $\mathcal{L}$ is defined by

$$\mathcal{L}U(p,t) \equiv \mathcal{L}^h U(p,t) + \tau^{-1}U(p,t),$$
$$\mathcal{L}^h U(p,t) \equiv d(p,t)U(p,t) - \sum_{p' \in \sigma'(p)} e(p',t)U(p',t).$$

Here $\sigma'(p) = \sigma(p) \setminus \{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p \in \omega^h$.

On each time level t, we make the following assumptions on the coefficients of the spatial operator $\mathcal{L}^h$:

$$d(p,t) > 0, \quad e(p,t) \geq 0, \quad p \in \omega^h, \tag{3}$$
$$d(p,t) - \sum_{p' \in \sigma'(p)} e(p',t) \geq 0, \quad p' \in \sigma'(p).$$

We also assume that the mesh $\overline{\omega}^h$ is connected. It means that for two interior mesh points $\tilde{p}$ and $\hat{p}$, there exists a finite set of interior mesh points $\{p_1, p_2, \dots, p_s\}$ such that

$$p_1 \in \sigma'(\tilde{p}), \quad p_2 \in \sigma'(p_1), \dots, \quad p_s \in \sigma'(p_{s-1}), \quad \hat{p} \in \sigma'(p_s). \tag{4}$$

On each time level t, introduce the linear problem

$$(\mathcal{L} + c)W(p,t) = f_0(p,t), \quad p \in \omega^h, \tag{5}$$