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CONVERGENCE AND COMPLEXITY OF ADAPTIVE FINITE ELEMENT METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study adaptive finite element approximations in a perturbation framework, which makes use of the existing adaptive finite element analysis of a linear symmetric elliptic problem. We analyze the convergence and complexity of adaptive finite element methods for a class of elliptic partial differential equations when the initial finite element mesh is sufficiently fine. For illustration, we apply the general approach to obtain the convergence and complexity of adaptive finite element methods for a nonsymmetric problem, a nonlinear problem as well as an unbounded coefficient eigenvalue problem.

Key Words. Adaptive finite element, convergence, complexity, eigenvalue, nonlinear, nonsymmetric, unbounded.

1. Introduction

The purpose of this paper is to study the convergence and complexity of adaptive finite element computations for a class of elliptic partial differential equations of second order and to apply our general approach to three problems: a nonsymmetric problem, a nonlinear problem, and an eigenvalue problem with an unbounded coefficient. One technical tool for motivating this work is the relationship between the general problem and a linear symmetric elliptic problem, which is derived from some perturbation arguments (see Theorem 3.1 and Lemma 3.1).

Since Babuška and Vogelius [3] gave an analysis of an adaptive finite element method (AFEM) for linear symmetric elliptic problems in one dimension, there has been much work on the convergence and complexity of adaptive finite element methods in the literature. For instance, Dörfler [10] presented the first multidimensional convergence result and Binev, Dehmen, and DeVore [5] showed the first complexity work, which have been improved and generalized in [5, 6, 9, 12, 13, 18, 19, 20, 21, 25], from convergence to convergent rate and complexity. For a nonsymmetric problem, in particular, Mekchay and Nochetto [18] imposed a quasi-orthogonality property instead of the Pythagoras equality to prove the convergence of AFEM while Morin, Siebrt, and Veeser [21] showed the convergence of error and estimator simultaneously with the strict error reduction and derived the convergence of the estimator by exploiting the (discrete) local lower but not the upper bound. In this paper, we can get the convergence and optimal complexity of nonsymmetric problems from our general approach directly. For a nonlinear problem, Chen, Holst and Xu [7] proved

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the convergence of an adaptive finite element algorithm for Poisson-Boltzmann equation while we are able to obtain the convergence and optimal complexity of AFEM for a class of nonlinear problems now. For a smooth coefficient eigenvalue problem, Dai, Xu, and Zhou [9] gave the convergence and optimal complexity of AFEM for symmetric elliptic eigenvalue problems with piecewise smooth coefficients (see, also convergence analysis of a special case [12, 13]). In this paper, we will derive similar results for an unbounded coefficient eigenvalue problem from our general conclusions, too. We mention that a similar perturbation approach was used in [9].

This paper is organized as follows. In Section 2, we review some existing results on the convergence and complexity analysis of AFEM for the typical problem. In Section 3, we generalize results to a general model problem by using a perturbation argument when the initial finite element mesh is sufficiently fine. In Section 4 and Section 5, we provide three typical applications for illustration, including theory and numerics.

2. Adaptive FEM for a typical problem

In this section, we review some existing results on the convergence and complexity analysis of AFEM for a boundary value problem in the literature.

Let $\Omega \subset \mathbb{R}^d (d \geq 2)$ be a bounded polytopic domain. We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms, see, e.g., [1, 8]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v \mid_{\partial\Omega} = 0\}$, where $v \mid_{\partial\Omega} = 0$ is understood in the sense of trace, $\| \cdot \|_{s,\Omega} = \| \cdot \|_{s,2,\Omega}$. The space $H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$, will also be used. Throughout this paper, we shall use C to denote a generic positive constant which may stand for different values at its different occurrences. We will also use $A \lesssim B$ to mean that $A \leq CB$ for some constant C that is independent of mesh parameters. All constants involved are independent of mesh sizes.

2.1. A boundary value problem. Consider a homogeneous boundary value problem:

$$
\left\{ \begin{aligned} Lu := -\nabla \cdot (\mathbf{A}\nabla u) &= f \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \partial\Omega, \end{aligned} \right. \tag{1}
$$

where $\mathbf{A} : \Omega \to \mathbb{R}^{d\times d}$ is piecewise Lipschitz over initial triangulation $\mathcal{T}_0$ and symmetric positive definite with smallest eigenvalue uniformly bounded away from 0 and $f \in L^2(\Omega)$.

Remark 2.1. *The choice of homogeneous boundary condition is made for ease of presentation, since similar results are valid for other boundary conditions* [6].

The weak form of (1) reads: find $u \in H_0^1(\Omega)$ such that

$$
a(u, v) = (f, v) \qquad \forall v \in H_0^1(\Omega), \tag{2}
$$

where $a(\cdot, \cdot) = (\mathbf{A}\nabla\cdot, \nabla\cdot)$. It is seen that $a(\cdot, \cdot)$ is bounded and coercive on $H_0^1(\Omega)$, i.e., for any $w, v \in H^1(\Omega)$ there exist constants $0 < c_a \leq C_a < \infty$ such that

$$
|a(w, v)| \leq C_a \|w\|_{1,\Omega} \|v\|_{1,\Omega} \quad \text{and} \quad c_a \|v\|_{1,\Omega}^2 \leq a(v, v) \ \forall w, v \in H_0^1(\Omega).
$$

The energy norm $\| \cdot \|_{a,\Omega}$, which is equivalent to $\| \cdot \|_{1,\Omega}$, is defined by $\|w\|_{a,\Omega} = \sqrt{a(w, w)}$. It is known that (2) is well-posed, that is, there exists a unique solution for any $f \in H^{-1}(\Omega)$.