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AN EXTENDED FINITE ELEMENT METHOD FOR 2D EDGE ELEMENTS

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Abstract. A new eXtended Finite Element Method based on two-dimensional edge elements is presented and applied to solve the time-harmonic Maxwell equations in domains with cracks. Error analysis is performed and shows the method to be convergent with an order of at least $\mathcal{O}(h^{1/2-\eta})$. The implementation of the method is discussed and numerical tests illustrate its performance.

Key words. Maxwell's equations, domains with cracks, XFEM, singularities of solutions

1. Introduction

EXtended Finite Element Methods (XFEM) have gathered much interest in the domain of fracture mechanics in the last ten years since they are able to simulate the behavior of the displacement field in cracked regions using a mesh that is independent of the crack geometry. Hence, a single mesh can be used in the simulation of crack propagation, avoiding remeshing at each time step as well as reprojecting the solution on the updated mesh. The XFEM methodology was introduced by Moës *et al.* in 1999 [23]. Its main idea consists in enriching the basis of a standard Lagrange Finite Element Method by a step function along the crack in order to take into account the discontinuity of the displacement field across the crack. Moreover, the singular behavior of the solution near the crack tip is taken into account exactly by the addition of some singular functions, similar to the idea of the singular function method of Strang and Fix (see [30]). In the initial method of Moës *et al.*, only the nodes of the element containing the crack tip are provided with crack tip enrichment and the method is shown to converge with a rate of $\mathcal{O}(\sqrt{h})$ as does a classical Finite Element Method in a cracked domain. To improve these results, several variants of the method have been developped. Béchet *et al.* [3] and Laborde *et al.* [20] introduce crack tip enrichment in a fixed area around the crack tip independent from the mesh size and get nearly optimal convergence rates. In [6, 8], a regular cut-off function with a mesh-independent support is used to localize the crack tip enrichment and a mathematical error analysis is performed. The XFEM methodology has been generalized to three-dimensional planar and non-planar cracks [32, 24, 15] as well as to new application fields as two-phase flows or fluid-structure interaction [9, 14]. To some extent, XFEM can be interpreted as a fictitious domain method as it has been pointed out in [17]. Indeed, both methods use meshes of a domain of simple geometry (like a rectangle or disk), and the shape of the physical domain Ω is taken into account in the variational formulation and the discretization spaces. This can be done by multiplying the shape functions of the finite element space by some appropriated function depending on the geometry of Ω: the characteristic function of Ω in the fictitious domain approach (see e.g. [5, 17]), a step function of Heaviside-type in XFEM. Usually, fictitious domain methods are based on a mixed formulation involving a Lagrange multiplier in order

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to deal with Dirichlet boundary conditions. In the original XFEM approach, the boundary condition is of Neumann-type and hence there is no need for a mixed formulation. We refer to [22, 31] for a generalization to Dirichlet-type conditions.

In this paper, we propose a new eXtended Finite Element Method based on two-dimensional edge elements that are commonly used in the discretization of the Maxwell equations (see [28] for the original paper by Nédélec and [25] for a general presentation in three dimensions). We focus on a simple model problem which describes the time-harmonic Maxwell equations in a translation invariant setting resulting in a two-dimensional problem. To our knowledge, it is the first time that an XFEM-type method is applied in the context of computational electromagnetism. Some fictitious domain methods for electromagnetic scattering problems have been proposed for example in [5, 12, 13], but in general the obstacle is given by some regular domain. The simulation of the electromagnetic field in the presence of cracks is important for instance in electromagnetic testing which is a special technique of non destructive testing in order to detect defects inside a conducting test object as metallic tubes or aircraft fuselage. The discretization of the electromagnetic field in the neighborhood of geometric singularities is quite difficult since the singular behavior is much stronger than in fracture mechanics: near a crack tip, the asymptotic behavior is as $r^{-1/2}$ for the electromagnetic field compared to $r^{1/2}$ for the displacement field in linear elasticity. On a geometry-dependent mesh, edge finite elements can handle these singularities provided the mesh is sufficiently refined near the singular points of the geometry [29]. In [4], a singular field method based on Lagrange Finite Elements has been presented for the time-harmonic Maxwell equations for different settings of the problem including regions with screens. Singularities of the electromagnetic field have been studied for polygons and Lipschitz polyhedra in [11, 27] and the analysis carries over to cracked domains.

As for the nodal XFEM, our eXtended Finite Element Method based on edge elements takes into account the *a priori* knowledge on the exact solution. On the one hand, the standard discretization space of edge elements is enriched by some basis functions multiplied with a step function of Heaviside-type in order to enable the tangential component of the approximate solution field to be discontinuous across the crack. On the other hand, appropriated singular fields are added to the discretization space in order to take into account the singular behavior near the crack tip. These singular fields are derived from the singular functions associated with the scalar Laplace operator.

The paper is organized as follows: in §2, we define the variational setting of the model problem and introduce the singular functions that describe the behavior of the solution field near the crack tip. We prove the decomposition of the solution into a regular and a singular part and give the global regularity of the regular part. In §3, we define the discretization space for the XFEM based on two dimensional edge elements and prove that the discretization is conforming in $\mathcal{H}(\mathrm{curl}, \Omega)$. Section 4 is devoted to the analysis of the XFEM interpolation error which yields a convergence rate of the method of at least $\mathcal{O}(h^{1/2-\eta})$ due to Céa's lemma. In §5, we discuss the implementation of the method and give a series of numerical results illustrating the theory and the performance of the method. Finally, we postpone in Appendix A some technical results concerning a vector extension operator involved in the error analysis in §4.

2. The model problem

In this paper, we focus on a simple model problem. We consider the time-harmonic Maxwell equations in a two dimensional cracked domain Ω. Eliminating