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JUMPS WITHOUT TEARS: A NEW SPLITTING TECHNOLOGY FOR BARRIER OPTIONS

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Abstract. The market pricing of OTC FX options displays both stochastic volatility and stochastic skewness in the risk-neutral distribution governing currency returns. To capture this unique phenomenon Carr and Wu developed a model (SSM) with three dynamical state variables. They then used Fourier methods to value simple European-style options. However pricing exotic options requires numerical solution of 3D unsteady PIDE with mixed derivatives which is expensive. In this paper to achieve this goal we propose a new splitting technique. Being combined with another method of the authors, which uses pseudo-parabolic PDE instead of PIDE, this reduces the original 3D unsteady problem to a set of 1D unsteady PDEs, thus allowing a significant computational speedup. We demonstrate this technique for single and double barrier options priced using the SSM.

Key words. barrier options, pricing, stochastic skew, jump-diffusion, finite-difference scheme, numerical method, the Green function, general stable tempered process.

1. Introduction

Every 6 months, the Bank for International Settlements (BIS) publishes an overview of OTC derivatives market activity. The report covers OTC derivatives written on credit, interest rate, currencies, commodities, and equities. Based on that as of the end of June 2009, the total notional amount outstanding in OTC derivatives across the above asset classes stood at about 604 trillion US dollars. Just over one twelfth of this figure is attributable to OTC derivatives on foreign exchange (FX), which includes forwards, swaps, and options. The notional in OTC FX options stands at $11 trillion, which is roughly fifty times the notional in exchange-traded FX contracts.

If one wishes to understand how FX options are priced, it becomes important to access OTC FX options data. Unfortunately, these data is not as readily available as its more liquid exchange-traded counterpart. As a consequence, almost all academic empirical research on FX options has focussed on the exchange-traded market. An exception is a paper by Carr and Wu [6] (henceforth CW), who examine OTC FX options on dollar-yen and on dollar-pound (cable). CW document an empirical phenomenon that is unique to FX options markets. Specifically, at every maturity, the sensitivity of implied volatility to moneyness switches signs over calendar time. This contrasts with the pricing of say equity index options, for which the sensitivity of implied volatility to moneyness is consistently negative over (calendar) time. Since practitioners routinely refer to the sensitivity of implied volatility to moneyness as skew, CW term this time-varying sensitivity "stochastic skew". Using time-changed Lévy processes, they develop a class of option pricing models which can accommodate stochastic skew. While their models can, in principle, be used to price any FX exotic, CW only develop their methodology for plain vanilla OTC FX options, which are European-style. In the OTC FX arena, there is a thriving market for barrier options, whose pricing is not covered by CW. The

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purpose of this paper is to show that barrier options can be efficiently priced in the stochastic skew model of CW. This model has 3 stochastic state variables, which evolve as a time-homogeneous Markov process. While Monte Carlo can be used to price barrier options in these models, this paper focusses on the use of finite difference methods. More specifically, we show that operator splitting can be used to price a wide variety of barrier options. In particular, we examine the valuation of down-and-out calls, up-and-out calls, and double barrier calls.

Our goal was to propose a splitting technique that could reduce the SSM model original 3D unsteady partial integro-differential equation (PIDE) to a set of simple equations. It turned out that this set contains just 1D unsteady partial differential equations (PDEs) that could be efficiently solved using well-known finite difference schemes. Providing second order of accuracy in time and space was the second important point to meet when building the corresponding numerical methods. Unconditional stability of the method was the third important criterion. So in this paper we present an algorithm, which consists of the following steps:

(1) Split the original 3D unsteady PIDE to 2 independent 2D unsteady PIDEs. This is an exact result with no splitting error.
(2) Split each 2D unsteady PIDE to 1D unsteady PIDE with no drift and diffusion and 2D unsteady PDE with mixed derivatives.
(3) Split the 2D unsteady PDE with mixed derivatives to a set of 1D unsteady PDE using technique of [17].
(4) Using our approach in [18], transform 1D unsteady PIDE with no drift and diffusion to a pseudo-parabolic PDE which then could be efficiently solved by using finite difference schemes for 1D unsteady parabolic PDEs.

We implement this algorithm, providing the second order of approximation in time and all space directions, both at each step of the algorithm and for the entire algorithm as well. Also, our scheme is unconditionally stable in time.

New results presented in the paper are:

- Proved a theorem on how to exactly split 3D PIDE derived under the SSM model into two 2D PIDEs.
- Used a new method to compute the integral term with a linear complexity. The foundation of the method were given in another our paper. And as applied to real multi-dimensional pricing problem it was never described before in the literature.
- Proposed a new approach which reduces solution of the above described 3D PIDE to the solution of a set of 1D unsteady PDEs. The algorithm is of second order of approximation over all space and time coordinates and unconditionally stable.
- Obtained new numerical results on prices of barrier options under the SSM model.

The rest of the paper is organized as follows. The next section lays out assumptions and notation of the SSM model and develops a PIDE that governs the arbitrage-free value of any barrier option under this model. It also discusses boundary conditions for the 3 types of barrier options that we cover. Section 3 validates that the matrix of second derivatives of the model is semi-positive definite. The next three sections show how operator splitting can be applied to the resulting boundary value problems. The penultimate section shows our numerical results, and the final section concludes.