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APPROXIMATE SIMILARITY SOLUTION TO A NONLINEAR DIFFUSION EQUATION WITH SPHERICAL SYMMETRY

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Abstract. In this article we construct an approximate similarity solution to a nonlinear diffusion equation in spherical coordinates. In hydrology this equation is known as the Boussinesq equation when written in planar or cylindrical coordinates. Recently Li et al. [8] obtained an approximate similarity solution to the Boussinesq equation in cylindrical coordinates. Here we consider the same problem in spherical coordinates with the prescribed power law point source boundary condition. The resulting scaling function has a power law singularity at the origin versus a logarithmic singularity in the cylindrical case.

Key Words. approximate solutions, similarity solutions, Boussinesq equation, nonlinear diffusion.

1. Introduction

Nonlinear diffusion equations appear in many branches of natural sciences and there are multiple methods to solve them, see e.g. [5, 7]. The Boussinesq equation appearing in hydrology is an example of a diffusion equation with a linear diffusivity [4]. Some mathematical properties of this equation in planar, cylindrical and spherical cases are analyzed in [1]. It is shown there that for certain initial and boundary conditions the problem can be reduced to a boundary value problem for a nonlinear ordinary differential equation using similarity transformations. Here we consider the zero initial condition, so as shown in [3] the solutions propagate with the finite speed. An additional complexity of the problem is due to the existence of a free boundary that must be found during the solution process. We note that exact solutions exist only for a very limited number of values of the parameters describing the behavior at the boundary. As a result numerical or approximate analytical techniques must be employed to obtain the solution. Often Shampine's method [11] is used to solve these problems numerically. There are many approximate analytical methods of solution. We shall list only some of them. In the planar case you can construct approximate polynomial solutions that satisfy certain properties of the true solution of the differential equation [10, 13, 14]. In the cylindrical case there is a logarithmic singularity at the inlet, so an approximate similarity solution must include a logarithmic term [8]. In this article we construct an approximate similarity solution to the Boussinesq equation in spherical coordinates. For a comprehensive review of the literature on approximate analytical solutions for hydrologic applications see [12].

Our main goal is to expand the results of [8] from the cylindrical case to the spherical case. While [8] deals with the construction of approximate similarity

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solution to the Boussinesq equation, here we use results of [1] to construct an approximate similarity solution when the Boussinesq equation is considered in the spherical setting and the flow is emanating from a point source. As emphasized in [1] only flux boundary conditions make physical sense.

The paper is structured as follows: In Section 2 we provide the mathematical formulation of the problem and obtain an approximate solution; In Section 3 we compare our approximate solution against the numerical solution; We summarize the findings in Section 4.

2. Theory

We consider the Boussinesq equation in the case of spherical symmetry

$$\theta_s \frac{\partial h}{\partial t} - \frac{K_s}{r^2}\frac{\partial}{\partial r}\left(r^2 h \frac{\partial h}{\partial r}\right) = 0, \tag{1}$$

where t is time and r is the distance from the point source. In hydrologic applications h is the pressure head, θ_s is the porosity and K_s is the conductivity. This equation also appears in nonlinear heat conduction (e.g. [15]) with the dependent variable being temperature.

A physically meaningful solution to this equation with specified flux at the origin is discussed in [1]. The case of a power-law flux (α in equations (2) and (3) below is a parameter related to the power) at the origin is special since it allows the reduction of (1) to a nonlinear ordinary differential equation. Using dimensional analysis analogous to Barenblatt's [1] and Li et al.'s [8] the following two groups of dimensionless variables can be formed

$$h = Mt^{\frac{2\alpha-3}{5}} f(\eta) \tag{2}$$

and

$$r = \eta N t^{\frac{\alpha+1}{5}}, \tag{3}$$

where f is a scaling function and η is a similarity variable. As in the cylindrical case [8], without loss of generality we can relate constants M and N by

$$MK_s = N^2\theta_s \tag{4}$$

and we can define N in terms of the position of the front r_0 where $h = 0$

$$r_0 = Nt^{\frac{\alpha+1}{5}} \tag{5}$$

which implies

$$f(1) = 0. \tag{6}$$

Substituting (2) and (3) into (1) we obtain an ordinary differential equation

$$\frac{\alpha+1}{5}\eta\frac{df}{d\eta} - \frac{2\alpha-3}{5}f + \frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2 f \frac{df}{d\eta}\right) = 0. \tag{7}$$

The exact solution exists only for $\alpha = 0$. The solution propagates with finite speed as we can see from (5). This result was first obtained in [1] and later a more rigorous proof was provided in [3]. So far we have not imposed any conditions on α.

At the origin the boundary condition for the flux q is specified as

$$q = -K_s 4\pi r^2 h \frac{\partial h}{\partial r} \quad \text{at} \quad r = 0. \tag{8}$$

Using (2) and (3) the above can be rewritten as

$$q = -K_s 4\pi N M^2 t^{\alpha-1}\left[\eta^2 f \frac{df}{d\eta}\right]_{\eta=0}. \tag{9}$$