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AN ERROR ESTIMATE FOR MMOC-MFEM BASED ON CONVOLUTION FOR POROUS MEDIA FLOW

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Abstract. A modification of the modified method of characteristics (MMOC) is introduced for solving the coupled system of partial differential equations governing miscible displacement in porous media . The pressure-velocity is approximated by a mixed finite element procedure using a Raviart-Thomas space of index k over a uniform grid. The resulting Darcy velocity is post-processed by convolution with Bramble-Schatz kernel and this enhanced velocity is used in the evaluation of the coefficients in MMOC for the concentration equation. If the concentration space is of local degree l, then , the error in the concentration is $O(h_c^{l+1} + h_p^{2k+2})$, which reflects the superconvergence of velocity approximation.

Key Words. Porous medium flow, characteristic methods, Bramble-Schatz kernel, convolution, convergence analysis

1. Introduction

Mathematical models used to describe porous medium flow processes in petroleum reservoir simulation, groundwater contaminant transport, and other applications lead to a coupled system of time-dependent nonlinear partial differential equations (PDEs) [1]. Conventional second-order finite difference or finite element methods (FDMs, FEMs) tend to yield solutions with spurious oscillations. In industrial applications, first-order upwind methods are commonly used to stabilize the numerical approximations, but they tend to generate excessive numerical diffusion and grid-orientation effect [1].

An MMOC-MFEM time-stepping procedure was proposed and successfully applied in the numerical simulation of miscible displacement processes in petroleum reservoir simulation [2], in which the MMOC [3] was used to solve the transport equation while an MFEM scheme [4, 5] was used to solve the pressure equation. The MMOC symmetrizes and stabilizes the transport equation, greatly reduces temporal errors, and so allows for large time steps in a simulation without loss of accuracy. The MFEM schemes generate an accurate approximation to the Darcy velocity, which are required for accurate approximation to the transport because advection and diffusion dispersion in the transport equation are governed by Darcy velocity. The MFEMs minimize the numerical difficulties occurring in finite difference or finite element caused by differentiation of the pressure and then multiplication by rough coefficients [6]. Numerical experiments showed that the MMOC-MFEM type of solution techniques is numerically very competitive [2, 7].

A delicate and rigorous mathematical analysis was conducted in [8], in which an optimal-order error estimate was proved for a family of MMOC-MFEM time stepping procedure for miscible displacement processes in two space dimensions.

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These analysis theoretically confirm the numerical strength and advantage of the MMOC-MFEM time stepping procedure. As noted by the authors [8], however, a primary shortcoming of these results is that they are value only if the Courant number of the numerical discretization tends to zero asymptotically. This constraint is numerically very restrictive and was not observed numerically. In fact, under this assumption, an optimal-order error estimate can be proved for a Galerkin FEM-MFEM time stepping procedure [9], in which a Galerkin FEM is used to solve the transport equation. Furthermore, in the context of a strongly advection-dominated equation, an explicit finite difference method would converge under this assumption [10]. This very restrictive constraint has become a standard assumption in subsequent analysis for the MMOC methods for coupled systems in porous medium flow [11].

The work about superconvergence approximation can be found in [12, 13, 14] for elliptic problems(or pressure equation). A study on superconvergence along Gauss lines for the coupled problem for porous media flow can be found in Ewing [15]. Douglas and Roberts [16] and Douglas and Milner [17] have derived a collection of error estimates for mixed finite element methods for second order elliptic equations. These results include errors in Soblev spaces of negative index and superconvergence approximation, via convolution with Bramble-Schatz kernel, to both the basic dependent variable (in our case, p) and the related gradient field (u). The partition $T_{h_p}$is composed of squares of side length h_p related to a uniform grid over Ω. Based on the idea of [16, 17], Douglas [18] introduced the method of Bramble-Schatz kernel to the miscible displacement problem. The resulting Darcy velocity based on the mixed method is post-processed by convolution with a Bramble-Schatz kernel and this enhanced velocity is used in the evaluation of the coefficient in the Galerkin procedure for the concentration. For a time-continuous scheme, Douglas [18] achieved the superconvergence result $O(h_c^{l+1} + h_p^{2k+2})$, which is obviously higher than the standard optimal error estimate $O(h_c^{l+1} + h_p^{k+1})$ for mixed methods.

The authors of [18] mentioned that it is necessary to discretize the time variable in order to obtain actual numerical information. It seems to be a straightforward task to get the time-stepping procedure and establish the corresponding error estimate, however, the constraint condition between the time step $\triangle t_c$ and the space partition size h_p such as $\triangle t_c = o(h_p)$ had to be required [9]. This condition means that a procedure is guaranteed to converge only if the Courant number tends to zero asymptotically, and it is even more restrictive than the CFL condition for an explicit scheme in the context of a strangely advection-dominated displacement process [10].

Wang [19, 20] proved an optimal-order error estimate for a family of MMOC-MFEM approximation to the coupled system of miscible porous medium flow, which holds even if the Courant number tends to infinity asymptotically. In this way, the estimates justify the numerical advantages and strength of the MMOC-MFEM time-stepping procedure.

The object of this work is to establish and analyze an MFEM-MMOC time stepping procedure for the above model. As in [18], we combine the post-processed Darcy velocity(via convolution with a Bramble-Schatz kernel function) with the evaluation of the concentration variable. The same order of superconvergence rate will be retained in the final error estimates. Here we emphasis what kind of constraint conditions is required for the convergence rate. By introducing a new induction hypothesis, the superconvergence can be derived and the constraint condition between $\triangle t_c$ and h_p will be lightened to be $\triangle t_c = O(h_p^{1/2+3\delta})$ for a small positive constant δ.