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A POSTERIORI ERROR ESTIMATE FOR STABILIZED FINITE ELEMENT METHODS FOR THE STOKES EQUATIONS

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Abstract. Computation with adaptive grid refinement has proved to be a useful and efficient tool in scientific computing over the last several decades. The key behind this technique is the design of a good a posterior error estimator that provides a guidance on how and where grids should be refined. In this paper, the authors propose and analyze a posteriori error estimator for a stabilized finite element method in computational fluid dynamics. The main contributions of the paper are: (1) an efficient a posteriori error estimator is designed and analyzed for a general stabilized finite element method, (2) a rigorous mathematical analysis is established for a theoretical justification of its efficiency and generality to other applications, and (3) some computational results with a comparison with other methods are presented for a computational justification of the proposed a posteriori error estimator.

Key Words. A posteriori error estimate, finite element methods, CFD, adaptive grid refinement

1. Introduction

Computation with adaptive grid refinement has proved to be a useful and efficient tool in scientific computing over the last several decades. The key behind this technique is the design of a good posteriori error estimator that provides a guidance on how and where grids should be refined. The goal of this manuscript is to propose and analyze a posteriori error estimator for a stabilized finite element method in computational fluid dynamics.

As was well-known in the analysis and employment of finite element methods in solving the Navier-Stokes equations, the inf-sup condition [3] has played an important role because it ensures a stability and accuracy of the underlying numerical schemes. A pair of finite element spaces that are used to approximate the velocity and the pressure unknowns are said to be stable if they satisfy the inf-sup condition. Intuitively speaking, the inf-sup condition is a measure that enforces a certain correlation between two finite element spaces so that they both have the required properties when employed for approximating the Navier-Stokes equations. It is well known that the two simplest elements P_1/P_0 (i.e., linear/constant) on triangle and Q_1/P_0 (i.e., bilinear/constant) on quadrilateral do not satisfy the inf-sup condition. Furthermore, they are known to be not stable, and therefore can not be trusted when employed in practical computation. In contrast, most known stable elements

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do not seem to be natural because their construction involves non-standard functions or polynomials which are not commonly used/implemented in popular engineering code packages. To eliminate the constraint of the inf-sup condition so that natural finite element spaces can be used, several stabilized finite element methods have been developed for the Stokes equations in the last two decades [12, 4, 13, 8]. These methods are gaining more and more popularity in computational fluid dynamics, and this paper is focused on a further study of them.

For simplicity, the study shall be conducted for the incompressible Stokes equation for which the stabilized method as proposed in [8] is employed. The main contributions of this paper are: (1) an efficient a posteriori error estimator is designed and analyzed for the said stabilized finite element method, (2) a rigorous mathematical analysis is established for a theoretical justification of its efficiency and generality to other model equations, and (3) some computational results with a comparison with other methods are presented for a computational justification of the proposed a priori error estimator.

It should be pointed out that a posteriori error estimators for the P_1/P_0 stabilized finite element methods have been studied by Kay and Silvester [14]. The error estimators as proposed in [14] are of residual type which is strongly related to the a priori error estimator to be presented in this paper. However, the result of this paper applies to finite elements of arbitrary order, and the grid refinement strategies are different from that of [14].

The research of one of the authors was heavily influenced by his connection with Dr. Richard Ewing, particularly in the area of fluid dynamics and grid local refinement techniques for finite element methods. In fact, the first time when this author learnt “grid local refinement” was through a lecture presented by Dr. Ewing in 1987 at the Institute of Mathematics and Its Applications, University of Minnesota. Dr. Ewing had been a long time advocator for promoting the use and research of grid local refinements in scientific computing. This paper was written in the memory of Dr. Ewing for his scientific stimulation and vision in the research of computational mathematics.

The paper is organized as follows. In Section 2, we review some notations and outline a stabilized finite element formulation for the Stokes equations. In Section 3, a posteriori error estimator is given and a theoretical justification for its reliability and efficiency is established. Finally in Section 4, we present some numerical experiments for three test problems with two different refinement strategies.

2. Preliminaries and the stabilized finite element method

For simplicity, we consider the homogeneous Dirichlet boundary value problem for the Stokes equations. This model problem seeks unknown functions $\mathbf{u} \in H^1(\Omega)^d$ and $p \in L^2(\Omega)$ satisfying

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is an open bounded domain in the Euclidean space $\mathbf{R}^d (d = 2, 3)$ with a Lipschitz continuous boundary $\partial\Omega$; $\mathbf{f}$ is a given function in $H^{-1}(\Omega)^d$; Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators respectively; $\nu > 0$ is a given constant representing the viscosity of the fluid. The given function/distribution $\mathbf{f} = \mathbf{f}(x)$ is the unit external volumetric force acting on the fluid at $x \in \Omega$. Without