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A UNIFORMLY OPTIMAL-ORDER ESTIMATE FOR BILINEAR FINITE ELEMENT METHOD FOR TRANSIENT ADVECTION-DIFFUSION EQUATIONS

QUN LIN, KAIXIN WANG, HONG WANG, AND XIAOBO YIN

Abstract. We prove an optimal-order error estimate in a weighted energy norm for bilinear Galerkin finite element method for two-dimensional time-dependent advection-diffusion equations by the means of integral identities or expansions, in the sense that the generic constants in the estimates depend only on certain Sobolev norms of the true solution but not on the scaling parameter ε. These estimates, combined with a priori stability estimates of the governing partial differential equations, yield an ε-uniform estimate of the bilinear Galerkin finite element method, in which the generic constants depend only on the Sobolev norms of the initial and right data but not on the scaling parameter ε.

Key Words. convergence analysis, Galerkin methods, integral identity, integral expansion, uniform error estimates

1. Introduction

Time-dependent advection-diffusion equations, whih arise in mathematical models of petroleum reservoir simulation, environmental modeling, and other applications [3, 12], admit solutions with moving fronts and complex structures, and present serious mathematical and numerical difficulties [9, 13]. Many numerical methods have been developed to solve these problems and corresponding optimal-order convergence rates were proved [1, 5, 6, 9, 13, 14, 15, 18, 24]. However, these estimates have the major drawback that the generic constants in these estimates depend inversely on the scaling parameter ε, and so could blow up as ε tends to zero.

ε uniform estimates have been sought to address these issues and some progress has been made [13]. In the context of time-dependent advection-diffusion equations, suboptimal- and optimal-order ε uniform estimates were obtained primarily for Eulerian-Lagrangian methods [2, 19, 20, 21, 22, 23]. In essence, an ε uniform estimate is somewhat a restatement that the estimate is independent of the Peclet number. Eulerian-Lagrangian methods combine the advection and capacity terms to reformulate the governing equation as a parabolic equation in the Lagrangian coordinate to carry out the temporal discretization [6, 16, 17]. Thus, the corresponding Peclet number is formally zero. This explains why ε uniform estimates were proved only for Eulerian-Lagrangian methods, even if these methods are much more complex to analyze.

In this paper we prove an ε-uniform optimal-order error estimate for the bilinear Galerkin finite element method for time-dependent advection-diffuson equations,

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which, to the best knowledge of the authors, is the first work of this type. The primary advantage of Galerkin method resides in the simplicity of the implementation of the method. Due to the use of a standard temporal discretization, the advection term must be analyzed with care to ensure the impact of the Peclet number to be handled properly.

The rest of this paper is organized as follows. In §2 we recall preliminary results that are to be used in the paper. In §3 we revisit the problem formulation and approximation properties that are to be used in the analysis. In §4 we prove ε-uniform optimal-order error estimate for the problem. In §5 we prove auxiliary lemmas. §6 contains concluding remarks.

2. Problem formulation and Preliminaries

We consider a time-dependent advection-diffusion equation in two space dimensions

$$\begin{array}{rclrcl} u_t + \nabla \cdot \big(\mathbf{v}(\mathbf{x},t)u - \varepsilon \mathbf{D}(\mathbf{x},t)\nabla u\big) & = & f(\mathbf{x},t), & (\mathbf{x},t) & \in & \Omega \times (0,T] \\ u(\mathbf{x},0) & = & u_o(\mathbf{x}), & \mathbf{x} & \in & \Omega. \end{array} \tag{2.1}$$

Here $\Omega = (a,b)\times(c,d)$ is a rectangular domain, $\mathbf{x} = (x,y)$, $\mathbf{v}(\mathbf{x},t) = (V_1(\mathbf{x},t), V_2(\mathbf{x},t))$ is a velocity field, $f(\mathbf{x},t)$ accounts for external sources and sinks, $u_o(\mathbf{x})$ is a prescribed initial data, $\mathbf{D}(\mathbf{x},t) = \big(D_{ij}(\mathbf{x},t)\big)_{i,j=1}^2$ is a diffusion-dispersion tensor that has uniform lower and upper bounds $0 < D_{min}|\boldsymbol{\alpha}|^2 \le \boldsymbol{\alpha}^T \mathbf{D}(\mathbf{x},t)\boldsymbol{\alpha} \le D_{max}|\boldsymbol{\alpha}|^2 < +\infty$ for any $\boldsymbol{\alpha} \in \mathbf{R}^2$ and $(\mathbf{x},t) \in \Omega \times [0,T]$. Here $0 < \varepsilon << 1$ is a parameter that scales the diffusion and characterizes the advection-dominance of Eq. (2.1), and $u(\mathbf{x},t)$ is the ε-dependent unknown function. Finally, problem (2.1) is closed by a boundary condition. Differential types of boundary conditions are considered in this paper, including a (homogeneous) Dirichlet boundary condition

$$u(\mathbf{x},t) = 0, \qquad (\mathbf{x},t) \in \Gamma \times [0,T] \tag{2.2}$$

where $\Gamma := \partial\Omega$ is the spatial boundary of Ω as well as a noflow boundary condition [3, 12] which describes an impermeable boundary and is characterized by $\mathbf{v}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) = 0$. On the noflow boundary Γ a homogeneous diffusive flux boundary condition is imposed

$$-(\mathbf{D}\nabla u)(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) = 0, \qquad (\mathbf{x},t) \in \Gamma \times [0,T]. \tag{2.3}$$

This type of boundary condition often arizes in applications such as petroleum reservoir simulation. Finally, a periodic boundary condition is also considered in this paper [12].

Let $W_p^k(\Omega)$ consist of functions whose weak derivatives up to order-k are p-th Lebesgue integrable in Ω, and $H^k(\Omega) := W_2^k(\Omega)$. Let $H_0^1(\Omega) := \Big\{v \in H^1(\Omega) : v(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma\Big\}$, and $H_E^m(\Omega)$ be the subspace of $H^m(\Omega)$, which consists of functions that are periodic with respect to the domain Ω. We also introduce the energy norm $\|f(\cdot,t)\|_{H_D^1(\Omega)} := \big(\int_\Omega \nabla f(\mathbf{x},t) \cdot \mathbf{D}(\mathbf{x},t)\nabla f(\mathbf{x},t) d\mathbf{x}\big)^{\frac{1}{2}}$.