INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 9, Number 2, Pages 460–478

A BUFFERED FOURIER SPECTRAL METHOD FOR NON-PERIODIC PDE

HUANKUN FU AND CHAOQUN LIU

Abstract. Standard Fourier spectral method can be used to solve a lot of problems with periodic boundary conditions. However, for non-periodic boundary condition problems, standard Fourier spectral method is not efficient or even useless. This work has developed a new way to use Fourier spectral method for non-periodic boundary condition problems. First, the original function is normalized and then a smooth buffer polynomial is developed to extend the normalized function domain. The new function will be smooth and periodic with both function values and derivatives, which is easy to be treated by standard FFT for high resolution. This method has obtained high order accuracy and high resolution with a penalty of 25% over standard Fourier spectral method, as shown by our examples. The scheme demonstrates to be robust. The method will be further used for simulation of transitional and turbulent flow.

Key words. Fourier spectral method, FFT, non-periodic PDE, buffer zone, high resolution.

1. Introduction

Spectral methods are a class of techniques used in applied mathematics and scientific computing to numerically solve certain partial differential equations (PDEs). One of the spectral methods is Pseudo-Spectral Method which utilizes the efficient algorithm of fast Fourier transform (FFT) to solve differential and pseudo-differential equations in spatially periodic domains. It has emerged as a powerful computational technique for the simulation of complex smooth physical phenomena, and its exponential convergence rate depends on the smoothness and periodicity of the function in the domain of interest. As is known, Fourier spectral method is a high order method and can be used to resolve the small length scales, which is particularly important for simulation of turbulence and acoustic problems because of its high resolution. Since it's inception in early 1970's, spectral methods have been extensively used to solve a lot of problems including turbulence. However, classical pseudo-spectral method imposes restriction on boundary conditions which must be periodic. Such a restriction cannot be applied to practical flows that usually have non-periodic boundary conditions. To overcome this problem, people did a lot of work, for example, changing the basis functions to be Chebyshev polynomials or other polynomials. In that way, one can get Chebyshev spectral method and so on, refer to [20], but they have lower resolution than the original trigonometry polynomials. Also we can use windows to treat the boundary conditions, then the physical solution near the boundaries is obtained by a regularized dewindowing operation, and on the inner domain, the unmodified equations are solved, refer to [9]. There are other works which also deal with parts of the problems, like [7] through [14].

As is known, even the function value itself (not derivative) is periodic on the boundary, the classical Pseudo- spectral method may still not work. Therefore, modifying the function and making the some orders derivatives of the function to be periodic on the boundary is very important. The effort in this work is focused on solving the above problems, trying to use the classical Fourier spectral method to get

Received by the editors June 9, 2011 and, in revised form, August 16, 2011.
2000 *Mathematics Subject Classification.* 35Q10, 65N30, 76D05.

the accurate derivatives of a function which is not periodic on boundary. Instead of using the classical Fourier spectral method directly for the problem, we first modify and extend the original function to get a new extended function (which is probably very different from the origin function), for which classical Fourier spectral method can be easily used. After getting the derivatives of the new function, we can easily recover the derivative of the original function.

The paper is organized as follows: Section 2 introduces the standard Fourier spectral method; Section 3 introduces our ideas and the new method; Section 4 gives some computational examples. The first part of Section 4 is for the derivative results. Here, we only focus on smooth functions, but we will continue to investigate for non-smooth functions. The other two parts of Section 4 give some preliminary results for the wave equation and Poisson equation.

2. Fourier Spectral Method

2.1. Fourier Interpolation. For a periodic sequence $\{x_n = 2\pi n/N, n = 0, 1, ...N\}$, the function $f(x)$ can be approximated by Fourier interpolation as:

$$I_N f = \sum_{k=-N/2}^{N/2} \frac{\tilde{f}_k}{\bar{c}_k} e^{ikx}, \tag{2.1}$$

here, $\bar{c}_k = \begin{cases} 1, & k = -N/2+1, ..., N/2-1; \\ 2, & k = \pm N/2. \end{cases}$ and $\tilde{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-ikx_j}$, $k = -N/2, ..., N/2$. The interpolation satisfies: $I_N f(x_n) = f(x_n), \quad n = 0, 1, ..., N-1$.

2.2. DFT. For a sequence $\{f(x_i)\}$, $i = 0, 1, ..., N-1$, the discrete Fourier transformation (DFT) is defined as:

$$\tilde{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f(x_j) e^{-2\pi ijk/N}, k = -N/2, ..., N/2-1. \tag{2.2}$$

The inverse transformation is:

$$f(x_j) = \sum_{k=-N/2}^{N/2-1} \tilde{f}_k e^{2\pi ijk/N}, j = 0, 1, ..., N-1. \tag{2.3}$$

2.3. Traditional Fourier spectral method for derivatives. Traditional Fourier spectral method for derivatives is based on the Fourier interpolation and use DFT/FFT to get coefficients of the DFT and then the original function derivatives. The original function is approximated by (2.1), and hence the derivative is:

$$f' = (I_N f)' = \sum_{k=-N/2}^{N/2} \frac{\tilde{f}_k}{\bar{c}_k}(ik) e^{ikx}$$

Therefore, if we want to get f' , we first use FFT to get the coefficients of DFT of f, and then multiply each of them with the corresponding number ik. After we perform the inverse DFT by FFT, the derivatives are available.

3. Buffered Fourier Spectral Method (BFSM)

The boundary condition for using standard Fourier spectral method is periodic, which is too restrictive. Even for simple functions like $y = x^2, (-1 \leq x \leq 1)$, with periodic boundary condition but non-periodic derivatives, the result is still a disaster, referring to the following sections. However, most of practical engineering problems have non-periodic boundary conditions. Therefore, it is very important to modify the Fourier spectral method so that it can be used for problems with non-periodic boundary conditions. This is the major purpose of the current work. This modified Fourier spectral method, called as buffered Fourier spectral method