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A HYPERBOLIC-ELLIPTIC MODEL OF TWO-PHASE FLOW IN POROUS MEDIA – EXISTENCE OF ENTROPY SOLUTIONS

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Dedicated to the memory of Magne S. Espedal

Abstract. We consider the flow of two-phases in a porous medium and propose a modified version of the fractional flow model for incompressible, two-phase flow based on a Helmholtz regularization of the Darcy phase velocities. We show the existence of global-in-time entropy solutions for this model with suitable assumptions on the boundary conditions. Numerical experiments demonstrating the approximation of the classical two-phase flow equations with the new model are presented.

Key Words. Porous media flow, conservation law, elliptic equation, weak solution, existence

1. The two Phase Flow Problem

Many geophysical and industrial processes like enhanced oil recovery and carbon dioxide sequestration involve the flow of two-phases, say oil and water, in a porous medium.

The variables of interest are the phase saturations s_w and s_o representing the saturation (volume fraction) of the water and oil phase respectively. We have the identity:

$$s_w + s_o \equiv 1. \tag{1.1}$$

Hence, we can describe the dynamics in terms of the saturation of either of the two-phases. We denote the water saturation as $s_w = s$ in the discussion below. Assuming a constant porosity ($\phi \equiv 1$), the two-phases are transported by [4]

$$(s_r)_t + \mathrm{div}_x(\mathbf{v}_r) = 0, \quad r \in \{w, o\}. \tag{1.2}$$

Here, the phase velocities are denoted by $\mathbf{v}_w$ and $\mathbf{v}_o$ respectively. In view of the identity (1.1), the two-phase velocities can be summed up to yield the *incompressibility* condition,

$$\mathrm{div}_x(\mathbf{v}) = 0, \quad \mathbf{v} = \mathbf{v}_w + \mathbf{v}_o. \tag{1.3}$$

The total velocity is denoted by $\mathbf{v}$.

The phase velocities in a homogeneous isotropic medium are described by the Darcy's law [4]:

$$\mathbf{v}_r = -\lambda_r \nabla_x p_r + \lambda_r \rho_r g \mathbf{k}, \quad r \in \{w, o\}. \tag{1.4}$$

Here, g is the constant acceleration due to gravity, $\mathbf{k}$ is the direction in which gravity acts and ρ_r is the (constant) density of the phase r. The quantity $\lambda_r = \lambda_r(s_r)$ is

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the phase mobility and p_r is the phase pressure. Assume that the *capillary pressure* i.e, $p_c = p_w - p_o$ is zero, we can sum (1.4) for both phases and obtain

$$\mathbf{v} = -\lambda_T(s)\nabla_x p + (\lambda_w \rho_w + \lambda_o \rho_o) g\mathbf{k}, \tag{1.5}$$

with $p = p_w = p_o$ being the pressure and $\lambda_T = \lambda_w + \lambda_o$ being the total mobility. Using (1.5), the gradient of pressure in (1.4) can be eliminated leading to

$$\mathbf{v}_w = \frac{\lambda_w(s)}{\lambda_T(s)}\mathbf{v} + \frac{\lambda_w(s)\lambda_o(s)}{\lambda_T(s)}(\rho_w - \rho_o)g\mathbf{k}.$$

Denoting the fractional flow function f as

$$f(s) = \frac{\lambda_w(s)}{\lambda_T(s)} = \frac{\lambda_w(s)}{\lambda_w(s) + \lambda_o(s)},$$

and the gravity function g as

$$g(s) = \frac{\lambda_w(s)\lambda_o(s)}{\lambda_T(s)}(\rho_w - \rho_o)g,$$

the saturation equation (1.2) for water can be written down as

$$s_t + \operatorname{div}_x(f(s)\mathbf{v} + g(s)\mathbf{k}) = 0. \tag{1.6}$$

Combining the saturation equation with the incompressibility condition (1.3) and the pressure equation, we obtain the evolution equations for two-phase flow in a porous medium:

$$\begin{aligned} s_t + \operatorname{div}_x(f(s)\mathbf{v} + g(s)\mathbf{k}) &= 0, \\ \operatorname{div}_x(\mathbf{v}) &= 0, \\ \mathbf{v} &= -\lambda_T(s)\nabla_x p + (\rho_w\lambda_w(s) + \rho_o\lambda_o(s))g\mathbf{k}. \end{aligned} \tag{1.7}$$

The above equations have to be augmented by suitable initial and boundary conditions.

The phase mobility $\lambda_w : [0,1] \mapsto \mathbb{R}$ is a monotone increasing function with $\lambda_w(0) = 0$ and the phase mobility $\lambda_o : [0,1] \mapsto \mathbb{R}$ is a monotone decreasing function with $\lambda_o(1) = 0$. Furthermore, the total mobility is strictly positive i.e, $\lambda_T \geq \lambda_* > 0$ for some λ_*.

The above equations are a hyperbolic-elliptic system as the saturation equation in (1.7) is a scalar hyperbolic conservation law in several space dimensions with a coefficient given by the velocity $\mathbf{v}$. The velocity can be obtained by solving an elliptic equation for the pressure p.

It is well known that solutions of hyperbolic conservation laws can develop discontinuities, even for smooth initial data, [8]. The presence of these discontinuities or shock waves implies that solutions of conservation laws are sought in a weak sense and are augmented with additional admissibility criteria or *entropy conditions* in order to ensure uniqueness.

As the two-phase flow equations involve a conservation law, we need to define a suitable concept of entropy solutions for these equations and show that these solutions are well-posed. The problem of proving well-posedness of global weak solutions of the two-phase flow equations (1.7) has remained open for many decades. The main challenge in showing existence is the fact that the velocity field $\mathbf{v}$ acts as a coefficient in the saturation equations. Although conservation laws with coefficients have been studied extensively in recent years, see [1, 11, 7, 2] and references therein, the state of the art results require that the coefficient is a function of bounded variation. Many attempts at showing that the velocity field $\mathbf{v}$ in (1.6) is sufficiently regular, for example is a BV function or has enough Sobolev regularity, have failed.