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CONVERGENCE OF MULTI-POINT FLUX APPROXIMATIONS ON GENERAL GRIDS AND MEDIA

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Abstract. The analysis of the Multi Point Flux Approximation (MPFA) method has so far relied on the possibility of seeing it as a mixed finite element method for which the convergence is then established. This type of analysis has been successfully applied to triangles and quadrilaterals, also in the case of rough meshes. The MPFA method has however much in common with another well known conservative method: the mimetic finite difference method. We propose to formulate the MPFA O-method in a mimetic finite difference framework, in order to extend the proof of convergence to polyhedral meshes. The formulation is useful to see the close relationship between the two different methods and to see how the differences lead to different strenghts. We pay special attention to the assumption needed for proving convergence by examining various cases in the section dedicated to numerical tests.

Key Words. Polygonal and polyhedral mesh, convergence, multi-point flux approximation, MPFA O-method, mimetic finite difference.

1. Introduction

When solving problems on a geological structure, one of the challenges that confronts numerical methods is grid deformation. In reservoir simulation a lot of work goes into making geological models. The models incorporate results both from direct measurements in the reservoir and statistical information from outcrops on land. The end result is a model that has a vertical resolution from around 20 cm to about a meter. Due to the size of the domain and the need to limit the size of the calculations, the vertical resolution of a typical simulation block lies between 5 and 20 meters. While the usual approach is to smooth out geological features, there is also a call for using more flexible grids. This includes grids that are heavily distorted as well as going beyond the standard quadrilaterals or hexahedra. The problem of grid deformation comes in addition to challenges such as anisotropies and discontinuities in the geological media due to layering and fractures.

One family of methods proposed for calculating the Darcy flow in reservoir simulation is the multi-point flux approximation (MPFA) methods. These have mostly been applied on quadrilaterals/hexahedra, though triangles and polygons have also been tested, see e.g. [2, 3, 18, 11]. While the MPFA methods have numerically been shown to converge, eg. [13, 24, 25], it has proved more difficult to prove the convergence analytically. Considering the theoretical convergence analysis, initial attempts sought how to reconstruct an interior vector field compatible with the fluxes which would recast the MPFA method as a mixed finite element method. An example of how the standard MPFA O-method is recast as a mixed finite element

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(MFE) method on triangulations can be found in [28, 18]. In [21, 29], examining quadrilaterals, an MPFA O-method which is derived from a mapping onto an orthogonal reference cell is analyzed. In a successive paper convergence is proved for the MPFA O-method on rough grids [22]. A new proof of convergence of the MPFA O-method on general grids has been presented by Agelas and Masson [7]. Here they show weak convergence of the gradient, but do not provide rate of convergence of the fluxes. Their proof is however valid on heterogeneous permeability fields and is not based on similarities with the MFE methods. In the recent papers by Bause, Hoffmann and Knabner, cf. [8], Matringe, Juanes and Tchelepi, cf. [30] and Ingram, Wheeler and Yotov, cf. [16], convergence of MPFA method is shown on the special cases of triangulations, parallelepipeds and hexahedra respectively. These papers are based on the relationship between the MFE method and the quadrature from [22]. In Klausen and Stephansen [20] the MPFA O-method in 2D is written as a mimetic finite difference (MFD) method by using the same quadrature from [22] but which then allows for general meshes. The paper includes a sketch of a convergence proof. The present paper is en extension of the ideas from [20], and the proof closely follows that of the articles [9, 23] on mimetic finite difference methods.

The aforementioned MFD method is known for its flexibility, as it can be defined on polygons or polyhedra and can be applied with grids presenting hanging nodes, see e.g. [1, 9]. In addition it converges even when the grid is heavily distorted or the anisotropy ratio of the permeability is high. What is less known is the close tie between the MFD method and the MPFA methods. Both classify for instance as raw field methods, in contrast to full field methods like the MFEM, cf. [17]. However, the differences are also important as they lead to different properties of the methods. In particular we note that the symmetry of the MFD method is what makes it so robust, as the mass matrix is tailored by the user to always be positive definite. On the other hand, this symmetry implies that the fluxes are globally coupled. Explicit local fluxes are useful in two-phase flow simulations, as the discretization for one-phase flow fluxes is easily updated by multiplying with the scalar mobility. It is in fact the discretization for explicit local fluxes that is the strength of the MPFA O-method. The price to pay is the loss of symmetry of the latter method, which impairs the convergence properties. We note that the standard implementation of the two methods is fundamentally different. While the MFD method involves solving a saddle point problem, the MPFA O-method is a finite volume scheme. We will however use the similarities between the two methods to prove convergence of the MPFA O-method.

We will show that a family of MPFA O-methods may be implemented as a MFD method. In fact, we will show that the MPFA O-method coincides with the local flux MFD method [23] when the latter uses a version of non-symmetric quadrature proposed in [22]. As the local flux MFD method is shown to converge on general polygons or polyhedra, this result applies to the MPFA O-method as well. We show how the analysis may be extended to include discontinuities in the permeability field as well.

The non-symmetry of the MPFA O-method poses an important limitation on the convergence proof in terms of what anisotropy ratios and grid deformations are permitted. However, the non symmetry of the method is what permits us to calculate local fluxes explicitly while still maintaining a limited stencil and obtaining convergence on rough grids. It should not therefore be seen as a fault of the method. It does mean though that extension to general polygons or polyhedra is not as straight forward as expected, as even too large grid deformations on quadrilaterals pose an