INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 9, Number 3, Pages 684–700

DISCUSSION OF DYNAMICS AND OPERATOR SPLITTING TECHNIQUES FOR TWO-PHASE FLOW WITH GRAVITY

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This paper is dedicated to the memory of Magne S. Espedal (1942–2010)

Abstract. We study the dynamics of two-phase flow with gravity and point out three different transport mechanisms: non-cyclic advection, solenoidal advection, and gravity segregation. Each term has specific mathematical properties that can be exploited by specialized numerical methods. We argue that to develop effective operator splitting methods, one needs to understand the interplay between these three mechanisms for the problem at hand.

Key Words. operator splitting, Helmholtz decomposition, porous media flow

1. Introduction

Numerical approximation of multiphase flow in heterogeneous reservoirs generally give rise to large systems of nonlinear equations that need to be solved to advance the solution forward in time. Developing a successful simulator therefore depends more on the robustness and efficiency of the nonlinear solvers than on the quality of the underlying discretization. This has led to widespread use of fully implicit formulations which promise unconditional stability. In practical simulations, however, robust implementations of fully implicit schemes must limit the length of the time step, depending on the complexity of the grid, the geology, fluid physics, discretization scheme etc. With increasingly large and complex reservoir descriptions, there is a growing demand for faster yet stable and predictable simulation technology. To achieve higher efficiency, solvers tend to exploit special features of the flow physics and possibly use some form of sequential operator splitting.

The key idea of operator splitting for an evolutionary problem is to divide the model equations into a set of subequations that each model some parts of the overall dynamics that can be conquered using a simpler or more effective solution method. An approximation to the evolutionary solution is then constructed by solving the subequations independently, in sequence or parallel, and piecing the results together. Formally, we want to solve a Cauchy problem of the form

$$\frac{dQ}{dt} + \mathcal{A}(Q) = 0, \qquad Q(0) = Q_0, \tag{1}$$

where $\mathcal{A}$ is an abstract and unspecified operator. The equation has the formal solution $Q(t) = \exp(-t\mathcal{A})Q_0$. Assume now that we can write $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_m$ in some natural way and that we know how to solve the subequations

$$\frac{dQ}{dt} + \mathcal{A}_j(Q) = 0, \qquad j = 1, \ldots, m \tag{2}$$

Received by the editors June 29, 2011.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.

more effectively that solving (1). Introducing a time step Δt, and setting $t_n = n\Delta t$, the operator splitting can formally be written as

$$Q(t_{n+1}) = e^{-t_{n+1}\mathcal{A}}Q_0 \approx \Big[e^{-\Delta t\mathcal{A}_m}\cdots e^{-\Delta t\mathcal{A}_2}e^{-\Delta t\mathcal{A}_1}\Big]Q(t_n). \tag{3}$$

Numerical methods are obtained by replacing the abstract operators $e^{-\Delta t\mathcal{A}_j}$ by numerical approximations. This way, one can combine numerical methods that have been developed to solve a particular class of evolutionary problems in a fairly straightforward manner, reusing specialized, highly efficient, and well-tested solvers. In particular, operator splitting enables easy replacement of one scheme with another scheme for the same elementary operator. Moreover, the use of operator splitting may also reduce memory requirements, increase the stability range, and even provide methods that are unconditionally stable.

One of the first operator splitting methods used within reservoir simulation, was the *alternating direction implicit* (ADI) method [30, 10], in which multi-dimensional flow problems were successfully reduced to repeated one-dimensional problems that could be effectively solved using the Thomas algorithm. Today, this method is seldom used. Instead, it is common to use operator splitting methods that split the computation of flow and transport into separate steps, e.g., methods such as IMPES, IMPSAT, sequential splitting, and sequentially fully implicit. Such splittings are essential for the development of specialized and highly efficient methods like multiscale pressure solvers [12] and streamline methods [9]. Operator splitting is used not only to separate flow and transport, but may also be used to separate different physical effects within a transport (or flow) equation. In particular, many previous studies have focused on splitting methods for parabolic transport equations designed to effectively capture the balance and interaction of viscous and capillary forces, see [15, 20] and references therein.

There are often several ways to decompose an evolution operator. A good starting point is to have effective and specialized solvers for parts of the problem, e.g., an effective pressure solver, an effective solver for advective flow, etc. Designing an optimal solution strategy, however, will also require a good understanding of how the different physical effects act together to form the overall dynamics of the problem so that one can: (i) optimize the operator decomposition into 'clean' subproblems that can be solved as effectively as possible, and (ii) efficiently piece together the resulting subsolutions without creating undesired artifacts in the approximate solution. Moreover, operator splitting can be used to accommodate the intuitive principle that each physical effect should (ideally) be evolved using its appropriate time constant.

In this paper, we discuss operator splitting for transport equations of the form

$$\phi\partial_t S + \nabla\cdot\big(f(S)\vec{v} + h(S,\vec{x})\vec{g}\big) = q, \tag{4}$$

involving only advective and gravitational forces. Our motivation for doing so is to understand how to utilize efficient advective solvers developed for the special case that the vector field is associated with potential flow and the hyperbolic characteristics of the system are always positive. The primary example is streamline simulation [9], but similar principles are used in methods for flow-based ordering [1, 23, 28]. In streamline simulation, the transport equation (4) is split into an advective and a gravity segregation part [18, 17, 6, 3]

$$\phi\partial_t S + \nabla\cdot\big(f(S)\vec{v}\big) = q, \qquad \phi\partial_t S + \nabla\cdot\big(h(S,\vec{x})\vec{g}\big) = 0. \tag{5}$$