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# TREATMENT OF TENSORIAL RELATIVE PERMEABILITIES WITH MULTIPOINT FLUX APPROXIMATION

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**Abstract.** Multi-phase flow in porous media is most commonly modeled by adding a saturation-dependent, scalar relative permeability into the Darcy equation. However, in the general case anisotropically structured heterogeneities result in anisotropy of upscaled parameters, which not only depends on the solid structure but also on fluid-fluid or fluid-fluid-solid interaction. We present a method for modeling of incompressible, isothermal, immiscible two-phase flow, which accounts for anisotropic absolute as well as relative permeabilities. It combines multipoint flux approximation (MPFA) with an appropriate upwinding strategy in the framework of a sequential solution algorithm. Different tests demonstrate the capabilities of the method and motivate the relevance of anisotropic relative permeabilities. Therefore, a porous medium is chosen, which is heterogeneous but isotropic on a fine scale and for which averaged homogeneous but anisotropic parameters are known. Comparison shows that the anisotropy in the large-scale parameters is well accounted for by the method and agrees with the anisotropic distribution behavior of the fine-scale solution. This is demonstrated for both the advection dominated as well as the diffusion dominated case. Further, it is shown that off-diagonal entries in the relative permeability tensor can have a significant influence on the fluid distribution.

**Key words.** porous media, two-phase flow, anisotropic medium, anisotropy, tensorial relative permeability, MPFA, upwinding, capillary pressure, gravity

## 1. Introduction

Multi-phase flow and transport phenomena in porous media are the governing processes in many relevant systems. An example for a natural system is the subsurface, considering for example the remediation of non-aqueous phase liquids or modeling of $CO_2$ storage scenarios (e.g. [15]). Biological systems can for example be found in the human body, where flow through the brain or in the lung can be modeled as flow through porous media (e.g. [36, 23, 20]), and there also exist many technical applications in which multi-phase flow through porous media is important (e.g. [9, 8]).

Flow and transport processes in permeable media occur on different spatial scales and are in general highly affected by heterogeneities. Usually, averaged equations applying an REV (Representative Elementary Volume) concept are used, where the most common model is the so-called Darcy equation.This model can be used for single phase flow as well as for multi-phase flow. Parameters of averaged equations usually directly (analytical methods, averaging methods, etc.) or indirectly (e.g. experiments, measurements, etc.) imply an upscaling of processes which occur on smaller scales. If upscaling methods are applied to flow in porous media with distinctive anisotropically structured block heterogeneities, a direction dependence of the upscaled large-scale parameters results. It is fairly common to assume and determine anisotropic absolute permeabilities on various scales. Anisotropic phase-dependent behavior is often neglected in the upscaling process. However, it has been observed at different scales that upscaling can also lead to phase-dependent anisotropic full-tensor effects (e.g. [34, 10, 18]). These can either be treated in a

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classical sense by deriving anisotropic phase-dependent parameters like phase and relative permeabilities respectively [33] or by upscaling strategies which are more closely linked to a certain discretization method and account for full-tensor effects by incorporating global effects into isotropic upscaled parameters (e.g. [14, 27]). We will further focus on the former. If the principal directions of an upscaled total permeability coincide with the directions of a cartesian computational grid, the extension of a basic finite volume scheme is quite obvious. In that case, it just has to be distinguished between the different grid directions. In all other cases, new numerical techniques have to be developed which are able to account for anisotropies which are represented by full tensor relative permeability functions, and which are largely independent of the choice of the grid (structured, unstructured). Moreover, upwinding strategies have to be revisited to account properly for the advection dominated behavior of multiphase flows.

In the following sections a mathematical model including the general case of anisotropic phase permeabilities is introduced and some mathematical as well as physical issues of the tensor properties of this parameters are discussed. It is important to choose a mathematical formulation which allows a numerical treatment which meets the challenges presented by the tensor properties. Further, a numerical scheme is developed which accounts for anisotropic behavior due to tensorial parameters in both the advective or gravity driven case as well as in the capillary dominated case. The scheme is based on multipoint flux approximation (MPFA) which has been derived for second order elliptic equations like Darcy's law [3, 17, 1]. There exist various types of MPFA methods where the most common one is the MPFA O-method. MPFA can be applied to unstructured grids [4] and in general shows good convergence properties for single-phase flow on quadrilateral grids [6, 32, 25, 13]. However, different kinds of MPFA methods differ with regard to convergence rates and monotonicity of the solution. Monotonicity of MPFA methods has been studied for example in [28, 29, 31]. The application of MPFA to multi-phase flow (extended Darcy) is straight forward as long as the relative permeabilities are described by scalar functions [3, 1]. In that case, the problem of evaluating the fluxes by MPFA is the same as for single-phase flow. However, if the relative permeabilities are tensors, the MPFA has to be extended to correctly account for the properties of this specific multi-phase flow regime. This applies to the simplified case neglecting capillary pressure and gravity, and becomes even more important for situations in which capillary pressure and gravity cannot be neglected. In particular, special attention is payed to a consistent upwinding strategy. The numerical method is tested on various examples, which are physically motivated and demonstrate the effects of anisotropic phase permeabilities as well as the capability of the model to account for this effects.

## 2. Mathematical model for two-phase flow

In the following, we describe our mathematical model for two-phase flow assuming immiscible and incompressible fluids. It is based on two conservation equations for mass, one for each of the fluid phases:

$$\phi \frac{\partial S_{\mathrm{w}}}{\partial t} + \nabla \cdot \mathbf{v}_{\mathrm{w}} - q_{\mathrm{w}} = 0, \tag{1}$$

$$\phi \frac{\partial S_{\mathrm{n}}}{\partial t} + \nabla \cdot \mathbf{v}_{\mathrm{n}} - q_{\mathrm{n}} = 0. \tag{2}$$

The wetting phase fluid is indicated by subscript w and the non-wetting phase fluid by subscript n, $S$ is the saturation, $\phi$ is the porosity of the porous medium and $q$