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A SINGULARLY PERTURBED CONVECTION–DIFFUSION PROBLEM WITH A MOVING INTERIOR LAYER

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Abstract. A singularly perturbed parabolic equation of convection-diffusion type with an interior layer in the initial condition is studied. The solution is decomposed into a discontinuous regular component, a continuous outflow boundary layer component and a discontinuous interior layer component. A priori parameter-explicit bounds are derived on the derivatives of these three components. Based on these bounds, a parameter-uniform Shishkin mesh is constructed for this problem. Numerical analysis is presented for the associated numerical method, which concludes by showing that the numerical method is a parameter-uniform numerical method. Numerical results are presented to illustrate the theoretical bounds on the error established in the paper.

Key Words. Singular perturbation, interior layer, Shishkin mesh.

1. Introduction

The solutions of singularly perturbed parabolic equations of convection-diffusion type typically contain boundary layers [1, 6, 16], which can appear along the boundary corresponding to the outflow boundary of the problem. Additional interior layers can form in the solution if either the coefficients, the inhomogenous term or the boundary/initial conditions are not sufficiently smooth [2]. In this paper, we examine a linear singularly perturbed parabolic problem with smooth data and an interior layer in the solution, which is created by artificially inserting a layer into the initial condition. This problem is motivated from studying a singularly perturbed parabolic problem of convection-diffusion type, with a singularity generated by a discontinuity between the boundary and initial conditions at the inflow corner. At some distance from this inflow corner, the solution is characterized by the presence of an interior layer moving in time along the characteristic of the reduced problem, which passes through the inflow corner. This paper formulates a related problem which captures this effect of an interior layer being transported in a convection-diffusion parabolic problem. Our interest is to design and analyse a parameter-uniform numerical method [6] for such a problem.

Parameter-uniform numerical methods for several classes of singularly perturbed parabolic problems of the form

$$-\varepsilon u_{xx} + au_x + bu + u_t = f, \quad (x,t) \in (0,1) \times (0,T],$$

with discontinuous coefficients (or discontinuous inhomogeneous term) have been constructed and analysed in [4, 15]. These methods are based on upwind discretizations combined with appropriate piecewise-uniform Shishkin meshes [6], which are aligned to the trajectory of the point where the data are discontinuous [15]. In

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[4, 15], it is assumed that the boundary/initial conditions are sufficiently smooth and compatible at the corners. The nature and width of any interior layers appearing in the solutions of these problems, is dictated mainly by the sign of the convective coefficient a either side of a discontinuity. In this paper, we examine a problem where the coefficient a is smooth and always positive, but the solution contains a strong interior layer, generated solely from the fact that the initial condition contains an internal layer.

Parameter-uniform numerical methods for singularly perturbed parabolic problems with a discontinuous initial condition have been examined by Shishkin et al. in a series of papers (see [7, 10, 11] and the references therein). Rather than using simple upwind finite difference operators on a piecewise-uniform mesh, Shishkin et al. use suitable fitted operator methods to capture the singularity in the neighbourhood of the discontinuity. In this paper, the initial solution is smooth, but contains an interior layer. We will see that it is not necessary to use a fitted operator method here, as a suitable Shishkin mesh combined with a standard upwind finite difference operator suffices to generate a parameter-uniform method.

This paper is structured as follows: In §2, we state the problem to be investigated. In §3, we employ a mapping [5] which is used to align the mesh to the location of the interior layer. The solution is decomposed into a sum of a regular component, a boundary layer component and an interior layer component. In §4, we examine the regular component and deduce parameter-explicit bounds on its derivatives. In §5 parameter-explicit bounds on the derivatives of the layer components are established, which are central to the design of a piecewise-uniform Shishkin mesh, given in §6. In §7, the associated numerical analysis is presented and in the final §8 some numerical results are given to illustrate the theoretical error bounds established in §7.

Notation. The space $\mathcal{C}^{0+\gamma}(D)$, where $D \subset \mathbf{R}^2$ is an open set, is the set of all functions that are Hölder continuous of degree γ with respect to the metric $\|\cdot\|$, where for all $\mathbf{u} = (u_1, u_2),\ \mathbf{v} = (v_1, v_2) \in \mathbf{R}^2$, $\|\mathbf{u} - \mathbf{v}\|^2 = (u_1 - v_1)^2 + |u_2 - v_2|$. For f to be in $\mathcal{C}^{0+\gamma}(D)$ the following semi-norm needs to be finite

$$\lceil f \rceil_{0+\gamma,D} = \sup_{\mathbf{u}\neq\mathbf{v},\ \mathbf{u},\mathbf{v}\in D} \frac{|f(\mathbf{u}) - f(\mathbf{v})|}{\|\mathbf{u} - \mathbf{v}\|^\gamma}.$$

The space $\mathcal{C}^{n+\gamma}(D)$ is defined by

$$\mathcal{C}^{n+\gamma}(D) = \{z : \frac{\partial^{i+j} z}{\partial x^i \partial y^j} \in \mathcal{C}^{0+\gamma}(D),\ 0 \le i + 2j \le n\},$$

and $\|\cdot\|_{n+\gamma},\ \lceil\cdot\rceil_{n+\gamma}$ are the associated norms and semi-norms. Throughout the paper, c or C denotes a generic constant that is independent of the singular perturbation parameter ε and of all discretization parameters.

2. Continuous problem

In this paper, we examine the following singularly perturbed parabolic problem: Find $\hat{u}$ such that

$$\text{(1a)}\quad \hat{\mathcal{L}}_\varepsilon \hat{u} := -\varepsilon \hat{u}_{ss} + \hat{a}(t)\hat{u}_s + \hat{u}_t = \hat{f}(s,t), \quad (s,t) \in Q := (0,1) \times (0,T],$$

$$\text{(1b)}\quad \hat{u}(s,0) = \phi(s;\varepsilon), 0 \le s \le 1,$$

$$\text{(1c)}\quad \hat{u}(0,t) = \phi_L(t),\ \hat{u}(1,t) = \phi_R(t),\ 0 < t \le T,$$

$$\text{(1d)}\quad \hat{a}(t) > \alpha > 0,$$