INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 9, Number 4, Pages 844–875

FINITE ELEMENT APPROXIMATIONS OF OPTIMAL CONTROLS FOR THE HEAT EQUATION WITH END-POINT STATE CONSTRAINTS

GENGSHENG WANG AND LIJUAN WANG

Abstract. This study presents a new finite element approximation for an optimal control problem (P) governed by the heat equation and with end-point state constraints. The state constraint set S is assumed to have an empty interior in the state space. We begin with building a new penalty functional where the penalty parameter is an algebraic combination of the mesh size and the time step. Based on it, we establish a discrete optimal control problem $(P_{h\tau})$ without state constraints. With the help of Pontryagin's maximum principle and by suitably choosing the above-mentioned combination, we successfully derive error estimate between optimal controls of problems (P) and $(P_{h\tau})$, in terms of the mesh size and time step.

Key words. Error estimate, optimal control problem, the heat equation, end-point state constraint, discrete.

1. Introduction

Let Ω be a bounded convex domain (with a smooth boundary $\partial\Omega$) in $\mathbb{R}^d$, $d = 1, 2, 3$. Let ω be an open subset of Ω and T be a positive number. We write Q for the product set $\Omega \times (0, T)$ and χ_ω for the characteristic function of the subset ω. Let $\langle \cdot, \cdot \rangle$ denote the inner product of the space $L^2(\Omega)$. Consider the following optimal control problem:

$$(P) \qquad \mathrm{Min} J(y, u)$$

over all such pairs $(y, u) \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$ that

$$\left\{ \begin{array}{lll} \partial_t y - \triangle y = \chi_\omega u & \text{in} & \Omega \times (0, T), \\ y = 0 & \text{on} & \partial\Omega \times (0, T), \\ y(0) = y_0 & \text{in} & \Omega \end{array} \right. \tag{1.1}$$

and

$$y(T) \in S.$$

Here, the initial data y_0 is a given function in $H_0^1(\Omega) \cap H^2(\Omega)$, the cost functional J is defined by

$$J(y, u) = \frac{1}{2} \int_0^T \int_\Omega (y - y_d)^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_\Omega u^2 \, dx \, dt,$$

the reference function y_d is taken from the space $H^1(0, T; L^2(\Omega))$, and the constraint set S satisfies the following conditions:

$(A1)$ $S \subset H_1^\perp$ is a convex and closed subset with a nonempty interior in $H_1^\perp$. Here, $H_1^\perp$ denotes the orthogonal subspace of H_1 in $L^2(\Omega)$, while H_1 is a subspace

Received by the editors June 11, 2011.

2000 *Mathematics Subject Classification.* 35K05, 49J20, 65M60.

This research was supported by the National Natural Science Foundation of China Under grants 10971158 and 1087114.

spanned by $f_1, f_2, \cdots, f_{n_0}$ with f_i, $i = 1, 2, \cdots, n_0$, being functions in the space $H_0^1(\Omega)$ and n_0 being a positive integer.

$(A2)$ The boundary of S, denoted by ∂S, is a C^1-manifold with one codimension in $H_1^\perp$. Furthermore, $\partial S = \{y \in H_1^\perp : F(y) = 0\}$, where $F \in C^1(H_1^\perp)$ holds the property that $F'(\xi) \in H_0^1(\Omega)$ whenever $\xi \in H_0^1(\Omega) \cap H_1^\perp$.

The purpose of this paper is to build a discrete approximating optimal control problem $(P_{h\tau})$ (where h and τ are the mesh size and time step, respectively), and then present an error estimate between optimal controls for those two problems. The main steps to reach the goals are as follows: We first set up a new penalty functional, where the penalty parameter is a suitable algebraic combination of the mesh size and the time step, then establish, with the aid of the penalty functional, a discrete approximating optimal control problem $(P_{h\tau})$ without state constraint, and finally, derive, with the help of the Pontryagin's maximum principle, an error estimate of optimal controls for those two problems. The main result of the paper can be approximately stated as: *the order of the L^2-error between optimal controls of the problems (P) and $(P_{h\tau})$ is $h^{\frac{1}{2}}$ whenever $\tau \approx O(h^2)$.*

In general, for parabolic equations, the study of optimal control problems with state constraints is much more difficult than the study of those without state constraints. This can be seen from the following points of view: (1) It is harder to show the existence of optimal controls for the problems with state constraints than those without state constraints. It may happen that a problem without state constraints has optimal controls while the same problem with a state constraint has no solution. (2) Some optimal control problems without state constraints hold the Pontryagin maximum principle, while the same problems with some state constraints do not have the Pontryagin maximum principle (see [5]). Therefore, to guarantee the problem (P) having optimal controls and holding the Pontryagin maximum principle, it is necessary to impose some conditions on S. It will be proved that when S satisfies the above-mentioned conditions $(A1)$ and $(A2)$, the problem (P) has a unique optimal control and holds the Pontryagin maximum principle. These two conditions are quite close to the finite codimensionality condition provided in [5].

The end-point state constraint is a very important kind of state constraints in the field of optimal controls for parabolic equations. To our surprise, the studies on error estimates for numerical approximations to optimal control problems for parabolic differential equations with end-point state constraint are very limited. Here we quote two related papers [11] and [12]. In [11], the authors studied numerical approximations of optimal controls for linear parabolic equations. The state constraint set in that paper was assumed to have interior points in the state space. In [12], the authors studied such a problem where the constraint set is a non-degenerate closed unit ball centered at the origin of the state space. An error estimate was established in [12]. Moreover, that estimate is better than what we have in this paper. However, the problem studied in the current paper properly covers the case in [12]. This will be seen from the following example:

Write $\{e_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$ for an orthonormal basis of $L^2(\Omega)$. Set $H_1^\perp = \text{span}\{e_{n_0+1}, e_{n_0+2}, \cdots\}$, where n_0 is a positive integer. Let $S \equiv \{y \in H_1^\perp : \|y\|_{L^2(\Omega)} \leq 1\}$. It is easy to check that S satisfies $(A1)$. Moreover, if we define $F : H_1^\perp \to (-\infty, +\infty)$ by $F(y) = \|y\|_{L^2(\Omega)}^2 - 1, \forall\, y \in H_1^\perp$, then $\partial S = \{y \in H_1^\perp : \|y\|_{L^2(\Omega)} = 1\} = \{y \in H_1^\perp : F(y) = 0\}$ and $F'(y) = 2y$, which imply that S satisfies $(A2)$.

Obviously, the above-mentioned S is a degenerate closed unit ball centered at the origin of the state space. Therefore, the framework of this paper properly covers