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ON ERROR ESTIMATES OF THE PENALTY METHOD FOR THE UNSTEADY CONDUCTION-CONVECTION PROBLEM I: TIME DISCRETIZATION

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Abstract. In this paper, the penalty method is proposed and discussed for the unsteady conduction-convection problem in two dimensions. In addition, we analyze its time discretization which is based on the backward Euler implicit scheme. Finally, the main results of this paper that optimal error estimates are obtained for the penalty system and the time discretization under reasonable assumptions on the physical data.

Key words. Unsteady conduction-convection problem, Penalty method, Time discretization, Optimal error estimates.

1. Introduction

In this paper, let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary $\partial\Omega$ or a convex polygon. Now we consider the following unsteady conduction-convection problem (cf. [3, 5]).

Problem (I) : Find u, p and T such that for $t_N > 0$,

$$(1)\qquad \begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \lambda jT, & (x,t)\in\Omega\times(0,t_N),\\ \text{div}\ u = 0, & (x,t)\in\Omega\times(0,t_N),\\ T_t - \lambda^{-1}\Delta T + u\cdot\nabla T = 0, & (x,t)\in\Omega\times(0,t_N),\\ u(x,t)=0,\ T(x,t)=0, & (x,t)\in\partial\Omega\times(0,t_N),\\ u(x,0)=0,\ T(x,0)=\varphi(x), & x\in\Omega, \end{cases}$$

where $u = (u_1(x,t), u_2(x,t))$ represents velocity vector, $p(x,t)$ the pressure, $T(x,t)$ the temperature, $\nu > 0$ the viscosity, $\lambda^{-1} > 0$ the thermal diffusivity, $j = (0,1)$ the two-dimensional unit vector, $\varphi(x,y)$ is the given function, t_N is the final time.

The unsteady conduction-convection Problem (I) is an important dissipative nonlinear system in atmospheric dynamics. It is the coupled equations governing viscous incompressible flow and heat transfer process [6, 22], where the incompressible flow is the Boussinesq approximation to the unsteady Navier-Stokes equations. There are many numerical methods have been studied on the conduction-convection problem (see [2, 5]) and many literatures (see [12, 13, 14, 15, 18]) are put into the construction, analysis and implementation for conduction-convection problem. Shen [19] firstly analyzed the existence uniqueness of approximation solution for steady conduction-convection equations with the Bernadi-Raugel element. Luo and his coworkers gave an optimizing reduced PLSMFE in [14] and a least squares Galerkin/Petrov mixed finite element method in [15]. Shi provided nonconforming

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mixed finite element method in [18]. An analysis of conduction natural convection conjugate heat transfer in the gap between concentric cylinders under solar irradiation was studied in [11], etc.

As we known, the velocity u, the pressure p and the temperature T are coupled together by the impressibility constraint "div$u = 0$" and two dissipative nonlinear equation, which make the system is difficult to solve by using the numerical methods. In order to overcome coupled problem, the penalty method as a popular pseudo-compressibility strategy which initially proposed by Courant [4] is popular used (see [7, 16, 17]). Temam [22] firstly applied it to the Navier-Stokes equations. Then, many works appeared on this subject. Shen [16] derived the optimal error estimates for the unsteady Navier-Stokes equations as follows:

$$\tau^{\frac{1}{2}}(t_n)\|u(t_n) - u_\varepsilon(t_n)\|_{L^2} + \tau(t_n)\|u(t_n) - u_\varepsilon(t_n)\|_{H^1} \leq C\varepsilon,$$

for $t_n \in [0, t_N]$, where $\tau(t_n) = \min\{1, t_n\}$, C is a general positive constant and $u(t_n), u_\varepsilon(t_n)$ are the solution of the Navier-Stokes equations and its penalty system, respectively. Recently, He [7] extended it to the finite element method. For the viscoelastic Oldroyd flow problem, Wang et al derived the optimal error estimates for the penalty system [23] and extended it to the fully discrete schemes [24]. This motivates our interest in solving more complicated problem by this method and we have investigated the unsteady conduction-convection problem. For the unsteady conduction-convection problem, the penalty method for Problem (I) is as follows.

Problem (II): Find $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$, p_ε and T_ε such that for $t_N > 0$,

$$\tag{2} \begin{cases} u_{\varepsilon t} - \nu\Delta u_\varepsilon + \tilde{B}(u_\varepsilon, u_\varepsilon) + \nabla p_\varepsilon = \lambda j T_\varepsilon, & (x,t) \in \Omega \times (0, t_N), \\ \text{div } u_\varepsilon + \frac{\varepsilon}{\nu} p_\varepsilon = 0, & (x,t) \in \Omega \times (0, t_N), \\ T_{\varepsilon t} - \lambda^{-1}\Delta T_\varepsilon + \tilde{B}(u_\varepsilon, T_\varepsilon) = 0, & (x,t) \in \Omega \times (0, t_N), \\ u_\varepsilon(x,t) = 0, \ T_\varepsilon(x,t) = 0, & (x,t) \in \partial\Omega \times (0, t_N), \\ u_\varepsilon(x,0) = 0, \ T_\varepsilon(x,0) = \varphi(x), & x \in \Omega, \end{cases}$$

where $0 < \varepsilon < 1$ is a penalty parameter,

$$\tilde{B}(u_\varepsilon, v_\varepsilon) = (u_\varepsilon \cdot \nabla)v_\varepsilon + \frac{1}{2}(\text{div}u_\varepsilon)v_\varepsilon \quad \text{and} \quad \tilde{B}(u_\varepsilon, T_\varepsilon) = u_\varepsilon \cdot \nabla T_\varepsilon + \frac{1}{2}(\text{div}u_\varepsilon)T_\varepsilon$$

is the modified bilinear term, $(\text{div}u_\varepsilon)v_\varepsilon$ and $(\text{div}u_\varepsilon)T_\varepsilon$ are introduced to ensure the dissipativity of Problem (II) as $(\text{div}u)v$ is introduced in the Navier-Stokes equations by Temam [21] to ensure the dissipativity of the Navier-Stokes equations. In this way, p_ε can be eliminated to obtain a penalty system that only contains $u_\varepsilon, T_\varepsilon$, which is much easier to solve than the original equations. Zhang and He have analyzed the penalty finite element for the stationary conduction convection problems [25] and the non-stationary conduction convection problems [26], they have given that, for all $t_n \in [0, t_N]$,

$$\tag{3} \begin{aligned} &\|u(t_n) - u_\varepsilon(t_n)\|_{L^2} + (\int_0^{t_n} \|u(t) - u_\varepsilon(t)\|_{H^1}^2 dt)^{\frac{1}{2}} + \|T(t_n) - T_\varepsilon(t_n)\|_{L^2} \\ &\qquad + (\int_0^{t_n} \|T(t) - T_\varepsilon(t)\|_{H^1}^2 dt)^{\frac{1}{2}} \leq C\sqrt{\varepsilon}, \end{aligned}$$

under the assumptions that the exact solutions are sufficiently smooth. When we consider the discrete problem for the penalty system (2), the estimate (3) is misleading. For instance, if the backward Euler scheme is applied to the penalized