INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 9, Number 4, Pages 907–927

EFFICIENT HOMOTOPY SOLUTION AND A CONVEX COMBINATION OF ROF AND LLT MODELS FOR IMAGE RESTORATION

FENLIN YANG, KE CHEN, AND BO YU

Abstract. The Rudin, Osher, and Fatemi model [20] (ROF) for image restoration has been extensively studied due to its edge preserving capability, but for images without edges (jumps), the solution to this model has the undesirable staircasing effect. To improve the model, Lysaker, Lundervold and Tai [14] (LLT) proposed a better second-order functional suitable for restoring smooth images but it is difficult to preserve discontinuities for non-smooth images. It turns out that results from convex combinations of ROF model and LLT model can preserve the main advantages of both models (see [16, 9]). In this paper, we first propose an applicable homotopy algorithm based fixed point method for the LLT model. We then propose two new variants of convex combination models. Numerical experiments are shown to demonstrate the advantages of these combination models and the robustness of our homotopy algorithm.

Key words. Image restoration, total variation, fourth-order PDE, fixed point method, homotopy method, convex combination.

1. Introduction

An observed image f can often become blurry and noisy during the formation, transmission or recording process for the original image u. The common additive degradation model is

$$f = Ku + \eta, \tag{1}$$

where η is an additive noise term and K is a known linear operator representing the blur (usually a convolution), the image is only corrupted by noise when K is the identity. The recovery of the original image from the observed image is an essential pre-processing phase for further image processing tasks such as edge detection, pattern recognition, and object tracking, etc.

The usual approach for image restoration solves the following constrained optimization problem:

$$\min_u \ R(u) \quad \text{subject to} \quad \|Ku - f\|^2 = \sigma^2. \tag{2}$$

This problem is naturally linked to the following unconstrained problem – the minimization of the total variation penalized least squares functional (see [20, 4, 24]):

$$\min_u \ \Big\{ J(u) = \alpha R(u) + \frac{1}{2}\|Ku - f\|^2 \Big\}. \tag{3}$$

Here $\|\cdot\|$ is the norm in $\mathbb{L}^2(\Omega)$ and α is a positive parameter controlling the trade-off between goodness of fit-to-the-data and variability in u. $R(u)$ is some functional which controls the regularity of u and ensures the solvability of the inverse problem

Received by the editors March 28, 2011 and, in revised form, July 13, 2011.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.
The research was supported by the National Natural Science Foundation of China (10671029) and the Research Fund for the Doctoral Programme of Higher Education (20060141029).

(1). Examples of regularization functionals that can be found in the literature [24, 26, 7, 2] include $R(u) = \|u\|,\ \|\triangle u\|,\ \|\nabla u\|$.

The total variation semi-norm proposed by Rudin, Osher, and Fatemi [20] (ROF) is one of the most effective regularization functionals for $R(u)$ which does not penalize discontinuities in u, and thus allows us to recover the edges of the original image. Its formula is

$$R_1(u) = TV(u) = \int_\Omega |\nabla u| dxdy = \int_\Omega \sqrt{u_x^2 + u_y^2} dxdy.$$

The corresponding Euler-Lagrange equation for (3) is

$$g_1(u) = -\alpha \nabla \cdot \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}} \right) + K^*(Ku - f) = 0, \tag{4}$$

with homogeneous Neumann boundary condition $\frac{\partial u}{\partial \vec{n}} = 0$, and $\vec{n}$ is the normal vector. Here β is a small positive parameter to avoid the denominator equals to zero. There are many fast methods for (4) (see [20, 22, 6, 17, 5, 8, 18]) up to now.

Although the ROF model yields very satisfactory results for removing noise while preserving edges, it suffers from the undesirable staircase effect for problems without sharp edges, namely the transformation of smooth regions (ramps) into piecewise constant regions (stairs). Some effort has been made to remedy this unfavorable property [15, 17, 19, 2, 21, 7, 10].

In [14], Lysaker, Lundervold and Tai (LLT) proposed a second-order functional as the regularization functional

$$R_2(u) = \int_\Omega |D^2 u| dxdy = \int_\Omega \sqrt{u_{xx}^2 + u_{xy}^2 + u_{yx}^2 + u_{yy}^2} dxdy.$$

The corresponding Euler-Lagrange equation for (3) using this $R_2(u)$ is

$$\begin{aligned} g_2(u) \quad = \quad & \alpha \left[\Big(\frac{u_{xx}}{|D^2u|_\beta} \Big)_{xx} + \Big(\frac{u_{xy}}{|D^2u|_\beta} \Big)_{yx} + \Big(\frac{u_{yx}}{|D^2u|_\beta} \Big)_{xy} + \Big(\frac{u_{yy}}{|D^2u|_\beta} \Big)_{yy} \right] \\ & + K^*(Ku - f) = 0, \end{aligned} \tag{5}$$

where β is a small positive parameter and $|D^2u|_\beta = \sqrt{u_{xx}^2 + u_{xy}^2 + u_{yx}^2 + u_{yy}^2 + \beta}$. It is known that the LLT model can recover smooth surfaces. However, there exist two major challenges in dealing with this model. One is to preserve jumps as done by the ROF model and the other is to get a more efficient solution method for (5) than the gradient descent.

To address the first challenge, one idea is to combine the models of ROF and LLT because we desire restoration properties of both models. Therefore, Lysaker and Tai [16] suggested a convex combination of the respective two solutions from (4) and (5). Specifically, with $w^0 = f$, a new iteration w^{k+1} is generated by the convex combination

$$w^{k+1} = \theta^k v^{k+1} + (1 - \theta^k) u^{k+1} \qquad k = 0, 1, 2 \cdots, \tag{6}$$

where v^{k+1} and u^{k+1} are respectively obtained by the kth time marching iteration of ROF model and LLT model with w^k as their old iteration. Here the parameter θ^k which is applied to control the combination depends on ∇w^k as follows:

$$\theta^k = \begin{cases} 1, & \text{if } |\nabla w^k| \ge c, \\ \frac{1}{2}\cos(\frac{2\pi |\nabla w^k|}{c}) + \frac{1}{2}, & \text{elsewhere,} \end{cases} \tag{7}$$