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ANISOTROPIC *hp*–ADAPTIVE DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS FOR COMPRESSIBLE FLUID FLOWS

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Abstract. In this article we consider the construction of general isotropic and anisotropic adaptive mesh refinement strategies, as well as *hp*–mesh refinement techniques, for the numerical approximation of the compressible Euler and Navier–Stokes equations. To discretize the latter system of conservation laws, we exploit the (adjoint consistent) symmetric version of the interior penalty discontinuous Galerkin finite element method. The *a posteriori* error indicators are derived based on employing the dual-weighted-residual approach in order to control the error measured in terms of general target functionals of the solution; these error estimates involve the product of the finite element residuals with local weighting terms involving the solution of a certain adjoint problem that must be numerically approximated. This general approach leads to the design of economical finite element meshes specifically tailored to the computation of the target functional of interest, as well as providing efficient error estimation. Numerical experiments demonstrating the performance of the proposed adaptive algorithms will be presented.

Key words. Discontinuous Galerkin methods, a posteriori error estimation, adaptivity, anisotropic *hp*–refinement, compressible flows

1. Introduction

The development of Discontinuous Galerkin (DG) methods for the numerical approximation of the compressible Euler and Navier-Stokes equations is an extremely exciting research topic which is currently being developed by a number of groups all over the world, cf. [1, 2, 3, 4, 6, 10, 11, 16, 20, 21, 22, 32, 33, 34], for example. DG methods have several important advantages over well established finite volume methods. The concept of higher-order discretization is inherent to the DG method. The stencil is minimal in the sense that each element communicates only with its direct neighbors. In particular, in contrast to the increasing stencil size needed to increase the accuracy of classical finite volume methods, the stencil of DG methods is the same for any order of accuracy, which has important advantages for the implementation of boundary conditions and for the parallel efficiency of the method. Moreover, due to the simple communication at element interfaces, elements with so–called hanging nodes can be easily treated, a fact that simplifies local mesh refinement (h–refinement). Additionally, the communication at element interfaces is identical for any order of the method, which simplifies the use of methods with different polynomial orders p in adjacent elements. This allows for the variation of the order of polynomials over the computational domain (p–refinement), which in combination with h–refinement leads to so–called hp–adaptivity.

Mesh adaptation in finite element discretizations should be based on rigorous *a posteriori* error estimates; for hyperbolic/nearly–hyperbolic equations such estimates should reflect the inherent mechanisms of error propagation (see [26, 27]). These considerations are particularly important when local quantities such as point values, local averages or flux integrals of the analytical solution are to be computed

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with high accuracy. In the context of aerodynamic flow simulations, it is of vital importance that certain force coefficients, such as the drag, lift and moment on a body immersed within a compressible fluid, are reliably and efficiently computed. Selective error estimates of this kind can be obtained by the optimal control technique proposed in [8] and [5] which is based on duality arguments analogous to those from the *a priori* error analysis of finite element methods. In the resulting *a posteriori* error estimates, the element-residuals of the computed solution are multiplied by local weights involving the adjoint solution. These weights represent the sensitivity of the relevant error quantity with respect to variations of the local mesh size. Since the adjoint solution is usually unknown analytically, it has to be approximated numerically. On the basis of the resulting *a posteriori* error estimate the current mesh is locally adapted and then new approximations to the primal and adjoint solution are computed. This feed-back process is repeated, for instance, until the required error tolerance is reached. In this way, optimal meshes, or in the *hp*–setting, optimal finite element spaces can be obtained for various kinds of error measures, where *optimal* can mean *most economical for achieving a prescribed accuracy TOL* or *most accurate for a given maximum number N_{max} of degrees of freedom.* This approach is quite universal as it can, in principle, be applied to almost any problem, as long as it is posed in a variational setting.

In this work, we consider the *a posteriori* error estimation and adaptive mesh design of the *hp*–version of the DG finite element method applied to compressible flows on general finite element spaces consisting of an anisotropic computational mesh with anisotropic polynomial degree approximation orders. Here, we shall be interested in the reliable and efficient approximation of certain target functionals of the underlying analytical solution of practical interest. In particular, (weighted) Type I *a posteriori* error bounds are derived, based on employing the dual-weighted-residual approach, cf. [5, 19, 28, 29], for example. Based on the *a posteriori* error bound we design and implement a series of adaptive algorithms to efficiently design the underlying finite element space. Inspired by our recent articles [12, 13], we consider adaptive mesh refinement algorithms based on utilizing anisotropic *h*–refinement, isotropic *hp*–refinement, and finally anisotropic *hp*–refinement. Within this latter strategy, once elements have been marked for refinement/derefinement, on the basis of the size of the local error indicators, the proposed adaptive algorithm consists of two key steps: (a) Determine whether to undertake *h*– or *p*–refinement/derefinement; (b) Select a locally optimal anisotropic/isotropic refinement. Step (a) is based on assessing the local analyticity of the underlying primal and adjoint solutions, on the basis of the decay rates of Legendre series coefficients; see our previous articles [16, 30, 29], together with [7]. Step (b) is based on employing a competitive refinement strategy, whereby the "optimal" refinement is selected from a series of trial refinements. This entails the numerical solution of a series of local primal and adjoint problems which is relatively cheap and fully parallelizable, cf. [13]. The work presented in this paper is a complete and improved account of our recent work announced in the book chapter [14].

This article is structured as follows. In Section 2 we introduce the three–dimensional compressible Navier–Stokes equations. Then, in Section 3 we formulate its discontinuous Galerkin finite element approximation, based on employing the adjoint consistent symmetric interior penalty method introduced in [23]. Then, in Section 4 we derive an error representation formula together with the corresponding (weighted) Type I a posteriori error bound for general target functionals of the solution. The error representation formula stems from a duality argument and