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DISCONTINUOUS GALERKIN METHOD FOR MONOTONE NONLINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper, we consider the incomplete interior penalty method for a class of second order monotone nonlinear elliptic problems. Using the theory of monotone operators, we show that the corresponding discrete method has a unique solution. The a priori error estimate in an energy norm is developed under the minimal regularity assumption on the exact solution, i.e., $u \in H^1(\Omega)$. Moreover, we propose a residual-based a posteriori error estimator and derive the computable upper and lower bounds on the error in an energy norm.

Key words. discontinuous Galerkin method, nonlinear elliptic problems, monotone, a priori error estimate, a posteriori error estimate.

1. Introduction

The discontinuous Galerkin (DG) methods were introduced in the early 1970s to solve first-order hyperbolic problems [17, 34, 38, 46]. Simultaneously, but quite independently, as non-standard schemes, they were proposed for the approximations of second-order elliptic equations [1, 41, 56]. Since the DG methods are locally conservative, stable and high-order methods, which can easily handle irregular meshes with hanging nodes and approximations that have polynomials of different degree in different elements, they have been studied extensively in the past several decades. We refer the reader to [2, 15, 16] for a comprehensive historical survey of this area of research, to [1, 11, 12, 23, 29, 42, 44, 45, 47, 50, 55, 56] and [52, 58] for the a priori error analysis of the DG methods for linear elliptic problems and optimal control problems, respectively.

Except for linear elliptic problems, some researchers have studied the a priori error estimates of the DG methods for the nonlinear elliptic problems. Houston, Robson and Süli [30] considered a one parameter family of *hp*-DG methods for a class of quasi-linear elliptic problems with mixed boundary conditions

$$-\nabla \cdot (\lambda(x, |\nabla u|)\nabla u) = f(x), \tag{1.1}$$

where the function λ satisfies the following monotone condition, i.e., there exist positive constants m_λ and M_λ such that

$$m_\lambda(t-s) \le \lambda(x,t)t - \lambda(x,s)s \le M_\lambda(t-s), \quad t \ge s \ge 0, \quad x \in \overline{\Omega}.$$

Using a result from the theory of monotone operators, the authors shown that the corresponding discrete method has a unique solution and derived the a priori error estimate in a mesh-dependent energy norm for $u \in C^1(\Omega) \cap H^k(\Omega), k \ge 2$, which is optimal in the mesh size and mildly suboptimal in the polynomial degree.

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Gudi, Nataraj, and Pani [27], Gudi and Pani[28] studied the *hp* local DG method and the *hp*-DG methods, respectively, for a class of quasilinear elliptic problems of nonmonotone type

$$-\nabla \cdot (a(x,u)\nabla u) = f(x), \quad \text{in} \quad \Omega, \tag{1.2}$$

proved the existence and uniqueness of the discrete solution and derived the a priori error estimates in a mesh-dependent energy norm and in the L^2-norm under the assumption $u \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. Bi and Ginting [4] considered the two-grid algorithm of the h-version DG method for (1.2) and derived the convergence estimates.

Recently, Gudi, Nataraj and Pani [26] analyzed a one parameter family of *hp*-DG methods for the following second order nonlinear elliptic boundary value problems

$$-\nabla \cdot \mathbf{a}(x,u,\nabla u) + \mathbf{a}_0(x,u,\nabla u) = f(x), \quad \text{in} \quad \Omega, \tag{1.3}$$

where the given functions $\mathbf{a}(x,y,z)$ and $\mathbf{a}_0(x,y,z)$ are twice continuously differentiable with all the derivatives through second order being bounded, and the matrix $\mathbf{a}_z(x,y,z)$ is symmetric and there exist two positive constants λ_1 and λ_2 such that

$$\lambda_1|\xi|^2 \le \xi^T \mathbf{a}_z(x,y,z)\xi \le \lambda_2|\xi|^2, \quad \forall x \in \Omega, \quad \forall y \in \mathbb{R}, \quad \forall z,\xi \in \mathbb{R}^2. \tag{1.4}$$

The authors developed the error estimate in the broken H^1-norm, which is optimal in h and suboptimal in p, using piecewise polynomials of degree $p \ge 2$, when the solution $u \in H^{5/2}(\Omega)$. We note that, in order to prove the existence and uniqueness of the DG solution, the assumptions $u \in H^{5/2}(\Omega)$ and $p \ge 2$ in [26] are necessary.

Additionally, we mention some related works in which the h-DG methods are used to solve the other nonlinear problems. We refer to [8] and [9] for (1.1) and monotone nonlinear fluid flow problems respectively, to [39] for nonlinear dispersive problems, to [43] for the nonlinear second-order elliptic and hyperbolic systems, to [48] for nonlinear non-Fickian diffusion problems and to [53] for nonlinear elasticity problems.

On the other hand, the a posteriori error estimates of DG methods have attracted many researchers' attention and some important results have been achieved. For the linear elliptic problems, we refer the reader to [3, 13, 19, 31, 32, 35, 36, 49] and the references therein for details. However, there are considerably fewer papers that are concerned with the nonlinear elliptic problems. To the best of our knowledge, there are only [7] and [33] in this direction. Bustinza, Gatica and Cockburn [7] used a Helmholtz decomposition of the error to derive a residual-based a posteriori error estimates in an energy norm of h-version local DG method for the nonlinear elliptic problems (1.2) in which the differential operators are strongly monotone. Similar technique has been used in [3, 13] for linear elliptic problems. Houston, Süli and Wihler [33] derived energy norm a posteriori error estimates of the *hp*-DG methods for (1.1) using the technique of the approximation of discontinuous finite element functions by conforming ones, which has been developed by some authors in the context of the h-DG methods in [35, 36, 37] and has been extended to *hp*-DG methods by [31, 33].

In this paper, we study the incomplete interior penalty method for the nonlinear elliptic problems that have the form (1.3) and are *monotonic* (specific assumptions on the functions $\mathbf{a}_i$, $i = 0,1,2$, where $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$, will be given in subsection 2.1). Our purpose in this paper is twofold. As a first task, we formulate the incomplete interior penalty method to the monotone nonlinear elliptic problems and prove that the form associated with this DG method is bounded, Lipschitz-continuous and strongly monotone. Then, using a result from the theory of monotone operators,