INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 1, Number 2, Pages 172–196

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Computing and Information

CUBIC SPLINE MESHLESS METHOD FOR NUMERICAL ANALYSIS OF THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS

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Abstract. The solution for the Navier-Stokes equations for incompressible steady state flow is presented using cubic spline (C2) continuous interpolation functions for the primary variables (velocities and pressure) on rectangular domains. The solution was explored for laminar flows with low, intermediate and high inertia effects. Two problems (Fluid squeezed between two plates and Wall-driven 2-D cavity flow) were solved using the presented scheme. Trial functions for velocities and pressure were chosen with cubic spline continuous interpolation functions on a rectangular grid that also satisfied the essential boundary conditions. The Galerkin weighted residual integrals were evaluated for the continuity and momentum equations. Using interpolation functions that satisfy the essential boundary conditions enabled the vanishing of any unknown boundary stress terms in the developed equations. The nonlinear equations were solved using an iterative technique. For low Reynolds number flows, coarse meshes were sufficient to reach convergence with very few iterations. For higher Reynolds number flows, a relatively finer mesh was necessary to reach a solution. The results show that cubic spline interpolation functions are suitable for solving the incompressible steady state flow Navier-Stokes equations using the Galerkin weighted residuals method. The chosen interpolation functions produced smooth continuous and differentiable results with relatively coarse meshes.

Key words. Navier-Stokes, Incompressible flow, C2 Interpolation functions, Cubic Splines

1. Introduction

Obtaining a stable numerical solution for the Navier-Stokes equations for non-linear incompressible, steady state flow has been the subject of extensive research throughout the years. Finite element (FE) analysis can be used, however, classical continuous Galerkin methods using C0 interpolation functions may be associated with problems such as pressure oscillations and numerical oscillations for high Reynolds number flows [16]. Modifications to traditional FE methods have been developed by many researchers to overcome these difficulties. Arnold et al. (1984)[1] have formulated a stable finite element for the solution of the Stokes equations (neglecting inertial effects, which contribute the non-linear terms). In this method, piecewise linear functions were used to discretize the velocity and pressure fields, while enriching the velocity profile with bubble functions. Multi-level FE methods have been used [11] in which the full non-linear Navier-Stokes equations are solved on a single coarse grid, and subsequent approximations are generated on a succession of refined grids by solving a linearized problem. More recently, this approach has also been combined with the stabilized mixed FE method [5]. Mixed FE methods as well as higher-order discontinuous Galerkin FE methods (DGFEM) have been readily used and are able to avoid spurious oscillations when appropriately defined [5, 13]. A recent approach by Thomas (2008)[18] combined the characteristic-based split method with a locally conservative Galerkin method to

Received by the editors May 20, 2010 and, in revised form, October 5, 2010.
2000 *Mathematics Subject Classification.* 76D05.
The authors would like to gratefully acknowledge the Natural Sciences and Engineering Research Council of Canada (NSERC) for providing funding for this project.

solve the Navier-Stokes equations for steady lid-driven cavity flow. The method involved an iterative procedure and made use of linear triangular elements, requiring a very fine mesh.

Alternatively, mesh free methods have been explored by many researchers because they allow the use of regular meshes on a simply shaped auxiliary domain [8] and overcome the difficulties associated with mesh generation [17]. Choe (2001)[3] has investigated the use of the Moving Least Squares Reproducing Kernel in the Galerkin formulation for the Navier-Stokes equations. Additionally, a mesh free method based on Weighted Extended B-splines (WEB-spline) has been used to formulate a stable solution to the Stokes and Navier-Stokes equations [8, 9, 10]. The element free Galerkin method has been used to solve the problem of fluid squeezed between two plates [17]. In this approach, three weight functions were examined including cubic spline, exponential, and rational functions. Solutions were obtained with a moderate number of nodes; however, inertial effects were neglected in the problem by only considering laminar Stokes flow.

Cubic spline functions have been available for decades [15], however, they have mainly been utilized for computer graphics and recently as interpolation functions for problems in structural mechanics [12, 7, 6]. In the current work, a meshless method is presented in which cubic spline functions are used for both the weight function as well as the interpolation function of the velocity and pressure in the solution of the Navier-Stokes equations. Cubic spline interpolation functions can provide a C2 continuous field for the variables sought. Compared to conventional FE analysis, the continuity in the variables (i.e. velocity and pressure) is improved from C0 to C2, giving a continuous and differentiable B-spline surface of order 3 for the variable of interest (velocity and pressure in the present case). The use of cubic spline interpolation functions in the solution of the Navier-Stokes equations for steady flow can improve upon previous methods by allowing a coarser mesh, including nonlinear inertial effects, and providing a continuous and differentiable solution surface for the velocity and pressure profiles. Continuity in the velocity and pressure allows for a more physically meaningful solution, and differentiability allows the solution to be directly compared with the original differential equations, which requires differentiation. In the case of the Navier-Stokes equations for incompressible flow, the use of C2 continuous interpolation functions can allow direct comparison of the numerical results with the governing differential equations, allowing a measure of the magnitude and location of residuals.

This work presents a numerical procedure by which the nonlinear steady state Navier-Stokes equations can be formed and solved using C2 spline interpolation functions for the velocities and the pressure variables. Two examples are investigated, which have been typically found in literature as validation test cases [18, 17, 4, 19]: 1. Fluid squeezed between two plates, 2. Wall-driven 2-D cavity flow. Continuous and differentiable velocity and pressure profiles are obtained for both examples. The location and magnitude of residuals are investigated for the solution surfaces as a measure of accuracy for the method. In addition, the influence of varying mesh sizes and flow conditions (by varying fluid density) is investigated. The results of the present method are compared with conventional FE analysis using a commercial software package (COMSOL Multiphysics v 3.4).

2. Methods

2.1. Cubic Spline Formulation. The cubic spline interpolation function $B_i(z)$ shown in Figure 1 can take the form [15]: