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# GREEN'S FUNCTION ESTIMATES FOR A SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEM IN THREE DIMENSIONS

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**Abstract.** A linear singularly perturbed convection-diffusion problem with characteristic layers is considered in three dimensions. Sharp bounds for the associated Green's function and its first- and second-order derivatives are established in the $L_1$ norm by employing the parametrix method. The dependence of these bounds on the small perturbation parameter is shown explicitly. The obtained estimates will be used in a forthcoming numerical analysis of the considered problem to derive a robust a posteriori error estimator in the maximum norm.

**Key words.** Green's function, singular perturbations, convection-diffusion, a posteriori error estimates

## 1. Introduction

Consider the following problem posed in the domain $\Omega = (0, 1)^3$:

$$L_{\boldsymbol{x}} u(\boldsymbol{x}) = -\varepsilon\, \Delta_{\boldsymbol{x}} u(\boldsymbol{x}) - \partial_{x_1}(a(\boldsymbol{x})\, u(\boldsymbol{x})) + b(\boldsymbol{x})\, u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega, \tag{1.1a}$$

$$u(\boldsymbol{x}) = 0 \qquad \text{for } \boldsymbol{x} \in \partial\Omega. \tag{1.1b}$$

Here $\varepsilon$ is a small positive parameter, and we assume that the coefficients $a$ and $b$ are sufficiently smooth (e.g., $a$, $b \in C^\infty(\bar{\Omega})$). We also assume, for some positive constant $\alpha$, that

$$a(\boldsymbol{x}) \geq \alpha > 0, \qquad b(\boldsymbol{x}) - \partial_{x_1} a(\boldsymbol{x}) \geq 0 \qquad \text{for all } \boldsymbol{x} \in \bar{\Omega}. \tag{1.2}$$

Under these assumptions, (1.1a) is a singularly perturbed elliptic equation, also referred to as a convection-dominated convection-diffusion equation. Its solutions typically exhibits sharp interior and boundary layers.

The Green's function for the convection-diffusion problem (1.1) exhibits a strong anisotropic structure, which is demonstrated by Figure 1. This reflects the complexity of solutions of this problem; it should be noted that problems of this type require an intricate asymptotic analysis [10, Section IV.1], [11]; see also [19, Chapter IV], [18, Chapter III.1] and [12,13]. We also refer the reader to Dörfler [3], who, for a similar problem, gives extensive a priori solution estimates.

Our interest in considering the Green's function of problem (1.1) and estimating its derivatives is motivated by the numerical analysis of this computationally challenging problem. More specifically, we shall use the obtained estimates in the forthcoming paper [5] to derive robust a posteriori error bounds for computed solutions of this problem using finite-difference methods. (This approach is related to recent articles [2, 15], which address the numerical solution of singularly perturbed

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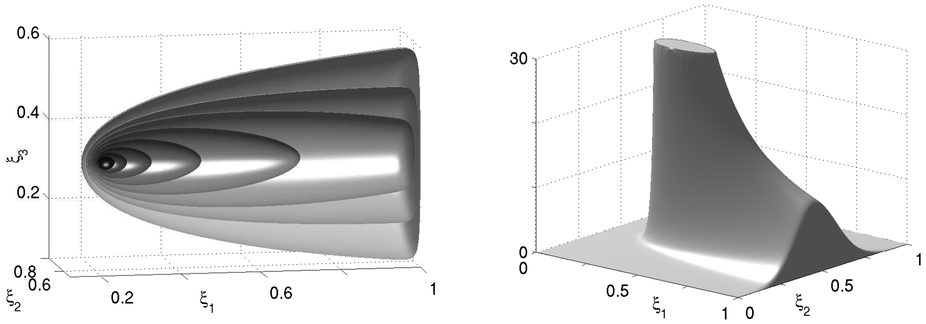


FIGURE 1. Anisotropy of the Green's function $G$ associated with (1.1) for $\varepsilon = 0.01$ and $\boldsymbol{x} = (\frac{1}{5}, \frac{1}{2}, \frac{1}{3})$. Left: isosurfaces at values of 1, 4, 8, 16, 32, 64, 128, and 256. Right: a two-dimensional graph for fixed $\xi_3 = x_3$.

equations of reaction-diffusion type.) More specifically, the basic idea is to rewrite the continuous residual of the computed solution $u^N$ in the form

$$L_{\boldsymbol{x}}[u^N - u] = \sum_{i=1}^{3} \partial_{x_i} F_i + \bar{f},$$

and then use stability properties of the differential operator, that follow from certain sharp bounds of the Green's function, to obtain a bound on the error $\|u^N - u\|_{\infty\,;\Omega}$. Here the computed solution $u^N$ is understood as a continuous function (if necessary, an interpolation of the discrete numerical solution has to be used). For more details, we refer the reader to Section 6. In a more general numerical-analysis context, we note that sharp estimates for continuous Green's functions (or their generalised versions) frequently play a crucial role in a priori and a posteriori error analyses [4, 9, 17].

The purpose of the present paper is to establish sharp bounds for the derivatives of the Green's function in the $L_1$ norm (as they will be used to estimate the error in the computed solution in the dual $L_\infty$ norm [5]). Our estimates will be *uniform in the small perturbation parameter* $\varepsilon$ in the sense that any dependence on $\varepsilon$ will be shown explicitly. Note also that our estimates will be *sharp* (in the sense of Theorem 2.2) up to an $\varepsilon$-independent constant multiplier. We employ the analysis technique used in [7], which we now extend to a three-dimensional problem. Roughly speaking, we freeze the coefficients and estimate the corresponding explicit frozen-coefficient Green's function, and then investigate the difference between the original and the frozen-coefficient Green's functions. This procedure is often called the parametrix method. To make this paper more readable, we deliberately follow some of the notation and presentation of [7], while some proofs that involve much computation and resemble the ones in [7] have been placed in a separate technical report [6].

The paper is structured as follows. In Section 2, the Green's function associated with problem (1.1) is defined and upper bounds for its derivatives are stated in Theorem 2.1, the main result of the paper. The corresponding lower bounds are then given in Theorem 2.2. In Section 3, we obtain and estimate the fundamental solution for a constant-coefficient version of (1.1) in the domain $\Omega = \mathbb{R}^3$. This